A Stochastic Model for the Time to Awaken in Response to a Fire Alarm

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ABSTRACT: A stochastic model for the time to awaken in response to a fire alarm is developed, based on a model originally developed by Ratcliff and Murdock in 1976 for analyzing two-choice decisions. It is mathematically identical to the classical "gambler's ruin problem." The model is fitted to two sets of data due to Bruck and her colleagues and validated by visual and statistical means. The model can be used to predict high quantiles of the awakening time, an essential component of the total time to escape, needed to evaluate the risk of death in a compartment fire.

KEY WORDS: fire alarms, occupant response, stochastic modeling, risk assessment.

INTRODUCTION

SMOKE DETECTORS ARE an essential part of fire fighting precautions taken to minimize the risk to life and property. Their purpose is to raise the alarm at an early stage of the fire so as to allow efficient evacuation of occupants and fire fighting. It is, however, well known that the speed at which various individuals awaken in response to a smoke detector alarm is very variable. It appears to depend on a multitude of factors which are usually not known in advance. As a consequence, it is possible to consider this response time as a random variable and to make use of its statistical properties in the context of a probabilistic risk analysis, as recommended, e.g., in Reference [1].

The most difficult aspect of the problem is the fact that it is necessary to estimate the upper tail of the distribution of response time, since it is those occupants who respond the most slowly who are at the most risk.

Direct estimation of the required probabilities from the data is not practical, because there is, unfortunately, a great dearth of data about the distribution of response time. To reasonably estimate the probability that a random variable exceeds some high value, when that probability is of the order of $1/n$, without any distributional assumptions, it is necessary to have a sample of size at least $10n$ (see

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Appendix A for proof). So, if we want to estimate the upper 1% quantile of a random variable, we need a sample of at least 1,000!

The most we can hope to achieve with the data available at present, without making any distributional assumptions, is to estimate means and standard deviations.

The alternative, which has been used extensively in other applications of probabilistic risk analysis and notably in structural engineering (see, e.g., Reference [2]), is to use a particular form of probability distribution for the random variable. The distribution usually contains one or more parameters which can be fitted to the observational data. One of the earliest (and probably most well known) applications of this idea was the modeling of the strength of brittle materials by W. Weibull in 1939, using what has become known as the Weibull distribution [3]. Much of the subsequent work in the area has been based on modeling the phenomenon being studied by an underlying stochastic process and then using a distribution derived from the stochastic process.

For the problem at hand, which is one of evaluating the response time of a human to a specific stimulus, it was possible to draw on a large body of recent work in psychology. After much consideration, it was decided to use a model for decision making based on the model proposed by Ratcliff and Murdock as early as 1976 [4]. This model has been continuously refined since then in a series of articles (see References [5–11]). The proposed distribution is indeed derived from an underlying stochastic process, namely the simple random walk.

In the simple random walk, a particle starts from some origin Z, and moves on a line at discrete points of time. At each point of time t it moves to the right by one unit with probability p and otherwise to the left by one unit (with probability $q = 1 - p$). In our particular application, the random walk ends in success if the particle reaches some point $A > Z$ and, in failure, if the particle reaches the point zero.

The observational data used was obtained by D. Bruck. One set was obtained in 1995 and is described in Reference [12]. The second was obtained in 1999 and is described in Reference [13]. The model proposed contains three parameters which are fitted to the observational data. The fit of the model is then evaluated by both visual and statistical means.

THE RANDOM WALK FOR TWO-CHOICE DECISIONS

The model used in this paper [4], is called a *memory retrieval theory* because it considers an item recognition task as testing a single probe item against a group of items in memory which has been designated as the memory search set. The retrieval process consists of comparing a collection of features of each element of the memory search set with the probe item. Ratcliff uses the term *relatedness* to describe the amount of match between them. He assumes that all information elicited can be mapped onto a unidimensional variable, that of relatedness.

In the application of the model to the response of subjects to a smoke detector alarm, the probe item is identified with the stimulus induced in the sleeping subject by the sound of the alarm. This stimulus is compared to the items of the memory search set. The comparison ends either by a match, in which case the subject wakes up, or by a nonmatch, in which case the subject continues to sleep.

The comparison process is described by Ratcliff as follows:

> Comparison of a probe to a memory-set item proceeds by the gradual accumulation of evidence, that is, information representing the goodness of match, over time. It is easiest to conceptualize the comparison process as a feature-matching process in which probe and memory-set item features are matched one by one. A count is kept of the combined sum of the number of feature matches and nonmatches, so that for a feature match, a counter is incremented, and for a feature nonmatch, the counter is decremented. The counter begins at some starting value Z, and if a total of A counts are reached, the probe is declared to match the memory-set item ($A - Z$ more feature matches than nonmatches). But if a total of zero counts are reached, an item nonmatch is declared. (Ratcliff [5, p. 63]).

It was already observed before Ratcliff's paper by Huesmann and Woocher [14] that the feature-matching, memory comparison process is mathematically identical ("isomorphic") to the classical "gambler's ruin problem" described in great detail in Feller's book *An Introduction to Probability Theory and Its Applications, Volume I* [15] in Chapter 14. We consider a gambler who wins or loses a dollar with probabilities p and $1 - p$ respectively. His initial capital is Z and his opponent's capital is $A - Z$. The game terminates either by a win, when the gambler's capital becomes A or by the gambler's ruin, when his capital reaches 0.

For various technical reasons, Ratcliff and his collaborators do not directly use the gambler's ruin model but rather an asymptotic approximation to it, the "diffusion process." They also use an extremely complex fitting procedure. This may be justified in the context of the large amount of data analyzed and a number of ancillary results sought. In the context of the limited amount of data available in this study, it was felt that a direct application of the gambler's ruin model would be sufficient.

It must be noted that whether one uses the original gambler's ruin model or the diffusion approximation, two strong assumptions are made:

1. The probability of a feature match p remains constant throughout the process.
2. The time between successive steps (i.e., the time to process a feature) also remains constant.

The uncanny fitting of the model to empirical data illustrated in Ratcliff's papers as well as the fit which will be described in the present paper are a vivid example of what Wigner called "the unreasonable efficiency of mathematics" [16]. In particular, the random walk model underlying the gambler's ruin problem has been successfully used in a wide variety of fields, from the early work of Einstein on the Brownian motion in 1905 to the Black-Scholes model of the pricing of stock options in the stock market in the early 1970s (for which Scholes was awarded a Nobel prize).

In the algorithm used in this paper, the total number of counts available, A, is chosen arbitrarily. The larger A, the better the fit can be achieved, but also the greater the computing effort required. (In the diffusion approximation used by Ratcliff, A is infinite.) The input from the data consists of:

1. The observed proportion p_Z of subjects who respond to the alarm.
2. For the subjects who respond, the mean and standard deviation of the response time.

From the given data, the algorithm estimates:

1. Z, the starting value of the feature match counter.
2. The probability p of a match.
3. The size of one step, say d.

From the formulae in Appendix B, the probability function of the time to a response can then be estimated and compared to the observed frequency.

THE 1995 DATA

In the experiment described in Bruck and Horasan [12] twenty-four sleeping young adults were exposed twice to a smoke detector alarm received at 60 dBA. Upon being awakened, time estimations, dream reports, alarm interpretations and computer reaction times were collected. Out of the 48 alarm presentations, 41 resulted in awakenings, i.e., a proportion $p_Z = 0.854$. The time to awaken data is given in the paper at intervals of 30 s.

To calculate the mean and standard deviation of the data, the observations were taken to occur at the midpoints of the intervals. To deal with the observations above 120, it was noted that the common ratio of the three previous observations was 0.5. By extrapolation, the three observations above 120 were therefore put at the mean of a geometric distribution with origin 120 s and ratio 0.5 for each interval of 30 s, namely at 150 s. On that basis, the mean time for all alarm presentations was 32.92 s with standard deviation 38.38 s. Thus, the coefficient of variation was 1.166. Upon fitting the model, as described in Appendix B, the estimated value of Z was 75 and that of p was 0.50. This corresponds to a theoretical proportion of awakenings $\hat{p}_Z = 0.75$. The length of a step, d, was 0.023 s.

Table 1 presents the observed and expected frequency distributions. The same data are illustrated in Figure 1.

A chi-square goodness of fit for the frequency distributions (see, e.g., Reference [17, p. 561]) was carried out as follows.

Table 1. 1995 Data: Comparison of observed and expected frequencies.

Midpoint	Interval	Observed	Expected
15	0–30	31	26.93
45	30–60	4	7.35
75	60–90	2	3.27
105	90–120	1	1.66
	>120	3	1.79

First, it was necessary to determine the number of degrees of freedom lost from the estimation. In fact, three parameters, namely *Z*, *p* and *d*, were estimated from the data. However, the data consisted of two parts:

1. The frequency distribution of the time to awakening, conditional on awakening.
2. The proportion of presentations not resulting in an awakening.

Thus, the number of degrees of freedom of the chi-square test must be reduced by only two to take into account estimation. Of course, a further degree of freedom must be taken away to take into account that the sum of the expected and observed values must be the same.

In addition, it is customary to aggregate cells with low expected numbers to en-

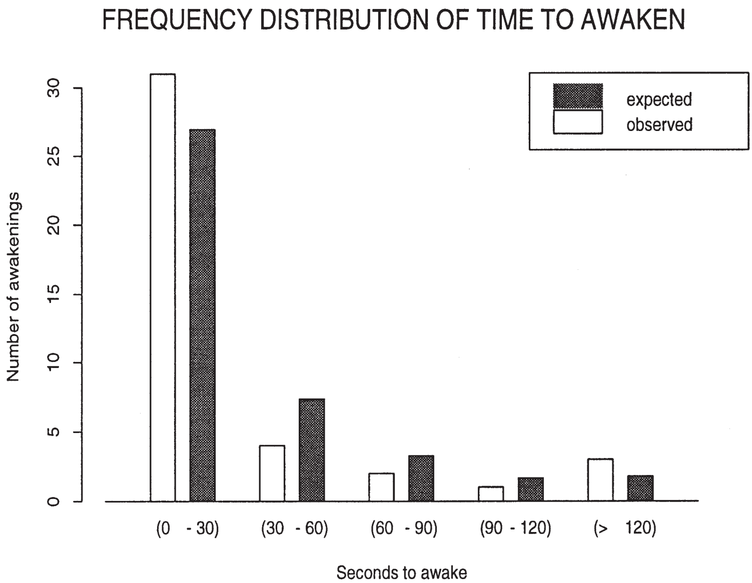

Figure 1. *Frequency distributions for 1995 data.*

sure that the expected numbers in each cell be at least 5. But, for our data, this would leave only three cells and make the test impossible.

We can, however, rely on the following result due to Yarnold [18] quoted in Kendall and Stuart [19, p. 457]: in the case of a simple null hypothesis, the minimum expected frequency may be as small as 5π where π is the proportion of classes with expectations less than 5. In the case of Table 1, π = 3/5 so that we can take the minimum expected frequency as 3. This can be achieved by aggregating the last two classes, leaving four classes and therefore one degree of freedom.

The value of chi-square is given by

$$\chi^2 = \frac{(31-26.93)^2}{26.93} + \frac{(4-7.35)^2}{7.35} + \frac{(2-3.27)^2}{3.27} + \frac{(4-3.45)^2}{3.45} = 2.72$$

This compares to $\chi^2_{0.05} = 3.843$ for one degree of freedom and indicates a very satisfactory fit.

A 95% confidence interval (Reference [17, p. 269]) for the theoretical proportion of awakenings was calculated, using the standard formula $\hat{p}_Z \pm 1.96 \times \sqrt{\hat{p}_Z(1-\hat{p}_Z/n}$. Since $\hat{p}_Z = 0.75$ and $n = 48$, the confidence interval turns out to be (0.6275, 0.8725), which contains the observed value 0.85.

Finally, the 99% quantile of the tail of the distribution of the time to awakening, conditional on awakening, was calculated, using the previously mentioned standard formula. The number of steps to the quantile turned out to be 8297, corresponding to 190.8 s.

THE YEAR 2000 DATA

The experiment described in Bruck and Bliss [13] followed a previous study that found that 85% of children did not reliably awaken to a standard hallway smoke alarm received at 60 dBA. It aimed at determining whether children will awake to a smoke alarm in their bedroom.

Twenty-eight children aged 6–15 years participated in the experiment and were exposed, on two different nights, in their own home, to an alarm which was received at 89 dBA. Sleep/wake behavior was determined by wrist actigraphy and confirmed by self-report questionnaires. The actigraphs yielded data at 16 s intervals and, from this, latencies to arouse were determined.

There were 56 alarm activations during which 34 children were awakened. Thus, the proportion of awakenings was 34/56 = 0.607. To calculate the mean and standard deviation of the data, the observations were taken to occur at the midpoints of the intervals. On that basis, the mean time for all alarm presentations was 41.41 s with standard deviation 48.53 s. Thus, the coefficient of variation was 1.17. Upon fitting the model, as described in Appendix B, the estimated

Table 2. 2000 Data: Comparison of observed and expected frequencies.

Midpoint	Interval	Observed	Expected
8	0–16	13	12.33
24	16–32	7	7.82
40	32–48	7	4.23
56	48–64	2	2.70
72	64–80	0	1.87
88	80–96	0	1.34
104	96–112	1	0.98
120	112–128	1	0.72
136	128–144	0	0.53
152	144–160	1	0.39
168	160–176	0	0.29
184	176–192	2	0.21
	>192	0	0.59

value of Z was 72 and that of p was 0.50. This corresponds to a theoretical proportion of awakenings $\hat{p}_Z = 0.72$. The length of a step, d, was 0.026 s.

Table 2 presents the observed and expected frequency distributions. The same data are illustrated in Figure 2.

For this second set of data, it was possible to carry out a chi-square test

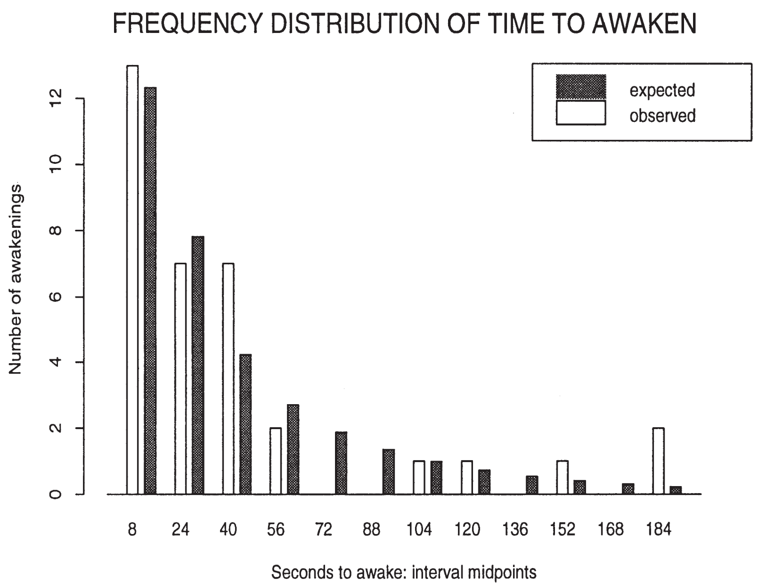

Figure 2. *Frequency distributions for 2000 data.*

with all expected numbers at least equal to 5. This was achieved by aggregating

1. Rows 3 and 4.
2. Rows 5 to 13.

The value of chi-square was given by

$$\chi^2 = \frac{(13-12.33)^2}{12.33} + \frac{(7-7.82)^2}{7.82} + \frac{(9-6.93)^2}{6.93} + \frac{(5-6.92)^2}{6.92} = 1.28$$

This compares to $\chi^2_{0.05} = 3.843$ for one degree of freedom and indicates a very satisfactory fit.

A 95% confidence interval for the theoretical proportion of awakenings was calculated as for the previous set of data. Since $\hat{p}_Z = 0.72$ and $n = 56$ the confidence interval turned out to be (0.6024, 0.8376), which contains the observed value 0.607.

Finally, the 99% quantile of the tail of the distribution of the time to awakening, conditional on awakening, was calculated using the previously mentioned standard formula. The number of steps to the quantile turned out to be 8554, corresponding to 222.4 s.

DISCUSSION

The data in the two experiments just described highlight the variability of smoke detector alarms in awakening sleeping persons. They are not, therefore, of much use in designing a fire protection system that will ensure a reasonable level of safety. Clearly, the proportion of subjects who do not wake up is unacceptably high. However, the random walk model developed would apply just as well if the proportion of awakenings was almost 100%. This can be achieved by increasing the probability of a match, p, to, say, 0.6, keeping the other parameters unchanged. The importance of being able to estimate high quantiles of the time to awakening will then become apparent.

In addition, it is clear that much larger data sets would be desirable to permit a more thorough examination of the model's accuracy. It would also be highly desirable to study how occupant characteristics such as age, as well as the effect of alcohol and drugs, will affect the speed of waking up in response to fire alarms. Once these characteristics are known, it should be possible to establish default parameters for each type of occupant population for use in actual design.

One factor that can be expected to decrease the risk of non-response is the existence of multiple cues. A generalization of the model presented in this paper to the situation where there are multiple cues, not occurring at the same time, will be presented in a forthcoming paper.

Furthermore, the analysis developed in this paper refers to only one component of the escape time: the time to awakening. The other components are usually taken to be the reaction time and the time of movement out of the danger area. The risk of fatalities is then determined by the interaction between the sum of these components and the time to onset of untenable conditions (see Reference [20]). It will be necessary to carry out much more experimental measurements as well as stochastic analyses of these additional components before a full-fledged evaluation of the risk of fatalities, based on more than guesses and anecdotal evidence, can be made.

CONCLUSION

A simple random walk model has been used to derive a probability of non-response as well as the distribution of the time to respond to a smoke detector alarm. The model is known to fit extremely well experimental data on two-choice decisions in a variety of contexts. When applied to two sets of data relating to the smoke detector alarm it also performed extremely well.

The proposed model can be used to estimate with considerable accuracy, for an alarm device:

1. The probability of non-response.
2. High quantiles of the distribution of response time.

NOMENCLATURE

A	maximum number of matches
d	size of one step
f_n	probability of a win at step n, conditional on winning
m	$1/p$
n	sample size in experimental data
N	sample size for estimation without distributional assumptions
N_0	number of sample values that exceed X_0
p	probability of a match
$\hat{p}$	estimator of p
p_Z	observed proportion of subjects who respond to alarms
$\hat{p}_Z$	estimator of p_Z
q	$1 - p$
q_Z	$1 - p_Z$
t	time
$\bar{T}$	observed mean response time
V	coefficient of variation of response time
$\hat{V}$	observed V
X	random variable

X_0 high value of X
Z starting value of match count
μ mean of time to win
ω p/q
π proportion of classes with expectations less than 5 in chi-square test
σ standard deviation of time to win
$\chi^2_{0.05}$ upper 5% percentile of chi-square

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APPENDIX A

We estimate the probability p, assumed small, that a random variable X exceeds a particular high value X_0, given only a random sample from the vari- able of size N, by $\hat{p} = N_0 / N$, where N_0 is the number of sample values that exceed X_0. The mean value of $\hat{p}$ is p and its standard deviation is $\sqrt{p(1-p)/N}$, which is approximately equal to $\sqrt{p/N}$ (since $1 - p$ is approximately unity). It follows that the coefficient of variation of $\hat{p}$ is approximately $1/\sqrt{p/N}$. If we require that the coefficient of variation be less than 30% it will be necessary for pN to be at least $1/0.3^2 = 11.1$. So, if p is approximately $1/n$, N will have to be at least $10n$ to have a reasonable coefficient of variation for the estimate.

APPENDIX B

In this appendix, formulae for the solution of the gambler's ruin problem will be given, based on Feller [B1], Chapter 14. Let p be the probability of a win at each step (i.e., a positive match) and $q = 1 - p$ the probability of loss. Let Z be the initial capital of the gambler (i.e., the initial value of the counter) and A the total capital available (i.e., the counter value for which the probe is declared to match the memory-set item). If the gambler's capital reaches A, the game ends in a win for the gambler and, if it reaches zero, the game ends in (ultimate) ruin (i.e., an item non-match is declared). We assume $0 < Z < A - 1$.

Let $\omega = p/q$, then the probability of winning p_Z is given by

$$p_Z = \frac{\omega^A - \omega^{(A-Z)}}{\omega^A - 1} \quad \text{if } \omega \neq 1$$

$$= \frac{Z}{A} \quad \text{if } \omega = 1$$

and the probability of (ultimate) ruin is given by $q_Z = 1 - p_Z$.

Explicit expressions for generating functions and probability functions are given in Section 5 of Chapter 14 in Feller's book [B1]. They refer to the time to ruin. But they can be easily modified to refer to the time to a win (i.e., a positive response to the alarm) by replacing p by q and Z by $A - Z$.

By differentiating twice the generating function given by Feller and using the appropriate boundary conditions, the following expressions for the mean μ and the standard deviation σ of the time to a win, given that the game ends in a win, can be derived.

Let the range of ν be $\nu = 1, \ldots, A - 1$ if A is odd. If A is even discard $\nu = A/2$. Let

$$s_\nu = \frac{1}{2(pq)^{1/2}\cos(\pi\nu / A)}$$

$$\rho_\nu = \omega^{(A-Z)/2}\frac{\sin(\pi(A-Z)\nu / A)\sin(\pi\nu / A)}{2A(pq)^{1/2}\cos^2(\pi\nu / A)}$$

then

$$\mu = 1 + \frac{1}{p_Z}\sum_\nu \frac{\rho_\nu}{s_\nu(s_\nu - 1)^2}$$

Furthermore, let

$$\mu_2^* = \frac{2}{p_z}\sum_\nu \frac{\rho_\nu}{(s_\nu - 1)^3}$$

then

$$\sigma^2 = \mu_2^* + \mu - \mu^2$$

Finally, let f_n be the probability of a win at step n, conditional on winning. Then

$$\begin{aligned} f_n &= 0 \qquad \text{when } n-(A-Z) \text{ is odd} \\ &= \frac{1}{p_Z}\left(\frac{2}{A}\right)(4pq)^{n/2}\omega^{(A-Z)/2}\sum_{\nu<A/2}\cos^{n-1}\frac{\pi\nu}{A}\sin\frac{\pi\nu}{A}\sin\frac{\pi(A-Z)\nu}{A} \qquad \text{(B-1)} \end{aligned}$$

otherwise.

As pointed out in the body of the paper, the parameter A is chosen arbitrarily. In the applications described in this paper, A was taken to be 100. From the data, the observed probability of non-response $\hat{p}_Z$ and the observed coefficient of variation of the response time $\hat{V}$, given that there was a response, are evaluated. The reason for using the coefficient of variation, i.e., the ratio of the standard deviation to the mean, is that it is independent of the length of one step, d. The parameters p (the probability of a match) and Z (the starting value) are estimated by fitting the corresponding values of p_Z and V to the observed values. In this paper, Z ranged from 1 to 99 in unit increments and p ranged from 0.01 to 0.99 in increments of 0.01. The fitting was carried out by minimizing $(V-\hat{V})^2 + (p_Z - \hat{p}_Z)^2$. Finally, the length of one step, d, was estimated as the ratio of the observed mean response time $\bar{T}$ to the mean number of steps μ corresponding to $V = \hat{V}$ and $p_Z = \hat{p}_Z$ in the model.

Once *Z, p* and *d* have been estimated, the probability function of response time can be estimated and compared to the observed frequencies, using Equation (B-1).

Reference

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