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Opinion Piece (2021)

Is NRR time-sensitive in measuring population replacement level?

By

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Summary: This technical piece introduces a concise debate on the implications of NRR (net reproduction rate), given that NRR only represents what we think it represents in a stable population, and no human population in practice is stable. The quantity NRR = 1 computed for any given year should not be treated as representing the attainment of replacement fertility in that year, because NRR = 1 represents replacement only in a stable population. A new measure Q_0 is proposed. Finally, the comment concludes that we either need a different measure for what we want or a different way of interpreting NRR.

The NRR is a frequently computed reproductive measure in demography that is often used for evaluating replacement-level fertility in a population. Alfred Lotka and Robert Kuczynski popularized it in the United states while understanding population growths and replacement levels. **[1,2].** In mathematical demography literature, NRR is denoted by R_0 **[1].** Let us understand first the technicalities of computing R_0 for an arbitrary year t. The standard formula to compute R_0 is

$$R_0 = \sum_{x=\alpha}^{x=\beta} \frac{B_x^t}{W_x^t}\frac{L_x^t}{l_0} - - - - - - - (1)$$

where B_x^t is the total number of female children born to the women of age x for the year t, W_x^t is the effective number of women of age x for the year t, and L_x^t is the life table population of women at age x obtained from single-age life table constructed for the year t with radix l_0 =100,000. Here α and β are the lower and upper limits of reproductive ages of women. The ratio B_x^t/W_x^t is also known as the age-specific female fertility rate at age x for the year t. The quantity L_x^t/l_0 is female life-table based survival rate from birth to the age x. The quantity L_x^t for a shorter age interval is often approximated as $(l_x^t + l_{x+1}^t)/2$. Here, l_x^t is the synthetic number of survivors at age x out of l_0. There are a few other ways to approximate L_x^t from l_x^t values, for example see G. King **[3].** The standard formula for NRR in (1) can be found in several textbooks, for example, see **[4].** When $R_0 = 1$ for the year t, we often say that the population has

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attained a replacement level of fertility in the year t. However, such a statement is only true when the population is stable.

In a population that is not stable, R_0 provides a futuristic population replacement value. NRR is therefore not sensitive enough to evaluate the population replacement levels in the year in which the data on the ratio B_x^t/W_x^t is collected.

The quantities B_x^t and W_x^t in (1) are computed from retrospective data. The quantity L_x^t/l_0 in (1) is computed from the life table available for the year t. A life table is computed from the age-specific mortality rates of the year t, and is a synthetic way to represent the course of survival. The quantity L_x^t/l_0 provides a futuristic value because a newly born girl in the year t would have some probability (say, $p(x)$) to attain the age x in the year $x+t$ for x value within α and β. This probability can never precisely be computed based on the data available in the year t. Hence, the futuristic value described creates uncertainty on the time to replace a woman with a girl child, and certainly, the replacement of the population is not happening in the year t.

We therefore need a new measure (Q_0) to understand retrospective replacement fertility. To overcome the uncertainty explained in the previous paragraph, even if we wait until the data becomes available to compute $p(x)$ in the future for every girl born in the year t to reach an age x in the year $x+t$ for x value within α and β, and multiply by the age-specific female fertility rates explained in (1), the resultant formula won't fit with the ideology of computation of R_0 of (1). Read the explanation below:

Let B_0^t be the total newly born girl babies in the year t such that $B_0^t = \sum_\alpha^\beta B_x^t$. Note that here α and β are the reproductive age range of women W_x^t. The survivors of B_0^t after several years would be part of the set, F_α^β in (3) described by

$$F_\alpha^\beta = \left\{F_\alpha^{t+\alpha}, F_{\alpha+1}^{t+\alpha+1}, \dots, F_\beta^{t+\beta}\right\} --- (3)$$

where F_j^{t+j} is the size of the female population of age j in the year $t+j$ who was born in the year t, for $j=\alpha,\alpha+1,\dots,\beta$. The probability of actual survival of girl babies to age j who were born in the year t will be

$$\frac{F_j^{t+j}}{B_0^t} \text{ for } j=\alpha,\alpha+1,\dots,\beta --- (4)$$

When we replace actual survival probability computed in (4) with L_x^t/l_0 in (1), that will lead to a quantity, say, Q_0 written as,

$$Q_0 = \sum_{x=\alpha}^{x=\beta} \frac{B_x^t}{W_x^t}\frac{F_x^{t+x}}{B_0^t} ------(5)$$

When $Q_0 = 1$ in the year $t+\beta$, we could retrospectively say that the population of the year t reached the replacement level of fertility in the year $t+\beta$. The expression of Q_0 of (5) can be written as

$$Q_0 = \sum_{x=\alpha}^{x=\beta} \frac{B_x^t}{W_x^t}\frac{F_x^{t+x}}{\sum_{x=\alpha}^{x=\beta} B_x^t} ------(6)$$

The structure of Q_0 expressed in (5) and (6) would be ideologically different from Lotka's or Kuczynski's principles of population growth using life tables. The

quantity Q_0 can be computed only retrospectively and its value does not indicate the average number of women replaced by girl child in the year t. Hence, R_0 computed for a population in the year t does not represent the replacement value of that population that has happened in the year t.

Implications:

If each of the girls out of B_0^t reaches the age α, then

$$B_0^t = F_\alpha^{t+\alpha},$$

else $B_0^t > F_\alpha^{t+\alpha}$. In general,

$B_0^t = F_j^{t+j}$ if the each of the girls of B_0^t reaches the age j for $j = \alpha, \alpha+1, \dots, \beta$, and $B_0^t > F_j^{t+j}$ if only a fraction of B_0^t girls reaches the age j for $j = \alpha, \alpha+1, \dots, \beta$. Hence, we will have non-decreasing terms of female populations as follows:

$$F_\beta^{t+\beta} \leq F_{\beta-1}^{t+\beta-1} \leq \cdots \leq F_\alpha^{t+\alpha} \leq B_0^t \text{ - - - - - } (7)$$

Two quantities Q_0 and R_0 will be identical only under the unrealistic situation of a stable population. Hence, even if $R_0 = 1$ in a year, due to varying population age-specific vital rates, we cannot conclude a replacement fertility has attained in that year. The time sensitivity of R_0 is lost. When the values of B_x^t, W_x^t and F_x^{t+x} are accurate, then the value of $Q_0 = 1$ indicates replacement level fertility of total females born in the year t has occurred several years after age-specific fertility data was collected. A numerical example with hypothetical data on girl child born, women in the year t and survivors of girl babies by various ages in the future. Refer to Table 1.

Table 1. Hypothetical example of computation of Q_0.

(1) Age	(2) B_x^t	(3) W_x^t	(4) F_x^{t+x}	(5) B_x^t/W_x^t	(6) F_x^{t+x}/B_0^t	(7) = (5) × (6)
15	5	11	25	0.455	0.926	0.421
22	7	9	23	0.778	0.852	0.663
32	6	7	22	0.857	0.815	0.698
37	5	6	19	0.833	0.704	0.586
45	4	5	18	0.8	0.667	0.533
Total=	27	38			$Q_0 =$	2.902

Note: (i) When the column (4) has the values 23, 19, 11, 10, 9, i.e., with a different set of future survivors out of 27 born in the year t, the resultant replacement fertility will be $Q_0 = 1.859$. (ii) **The value of F_x^{t+x} in the table can also be considered as effective number of women at age x. Smaller adjustments can be further done to obtain effective number of women with a linear or non-linear assumption on the number of years spent by a group of surviving women. (ii) In the NRR formula given the equation (1) one could also consider B_x^t as the total number of children instead of only female children considered and multiply the equation by the fraction (1/2.05) where 2.05 (=1+1.05) is the sex ratio at birth, i.e. on an average 1 female child is born per 1.05 male children. In that case we will consider life table values for the total population. Similar adjustments can be made in computation of Q_0.**

Q_0 from future births data of W_x^t:

Suppose we consider 38 women who have delivered girl babies at various ages as shown in Table 1. The women $W_{15}^{t} = 11$ who have delivered 5 girl babies in the year t have a possibility to deliver girl babies until and if they reach the age β. Such possibilities also exist for the women W_{22}^{t}, W_{32}^{t}, W_{37}^{t} and W_{45}^{t}. The average number of girl babies, say, $\boldsymbol{Q'_0}$ who will be replacing $\sum_{\alpha}^{\beta} W_x^t$ by the time all the women of $\sum_{\alpha}^{\beta} W_x^t$ completes their child bearing is expressed in (8) as

$$\boldsymbol{Q'_0} = \frac{\sum_{s=t}^{s=t+\beta} \boldsymbol{B}\left(\boldsymbol{s}: \sum_{\alpha}^{\beta} W_x^t\right)}{\sum_{\alpha}^{\beta} W_x^t} - - - - - - - - (\mathbf{8})$$

In (8), the quantity $\boldsymbol{B}\left(\boldsymbol{s}: \sum_{\alpha}^{\beta} W_x^t\right)$ is the total number of girl babies born for survivors of the women $\sum_{\alpha}^{\beta} W_x^t$ until they completed childbearing period. A numerical visualization of $\boldsymbol{Q'_0}$ is provided in Figure 1.

Suppose we have the data on the number of women who have completed the childbearing during the year t and their total female child born. Then the average number of girl children replacing the mothers who have completed their reproductive period during the year t can be computed by the quantity, say, Q''_0, which is expressed in (9) as

$$\boldsymbol{Q''_0} = \frac{\sum_{c=\alpha}^{c=\beta} B_c^t}{\sum_{c=\alpha}^{c=\beta} W_c^t} - - - - - - - - (\mathbf{9})$$

In (9), B_c^t is the total number of children born to the women who completed their childbearing at the age c and W_c^t is the number of women in the year t who have completed their childbearing at the age c. For example, if there are 77 women who have completed their child bearing in the year t and these 77 have delivered collectively 85 girl babies in their life, then $Q''_0 = \frac{85}{77} = 1.103$.

Q_0 from women who completed childbearing in the year t:

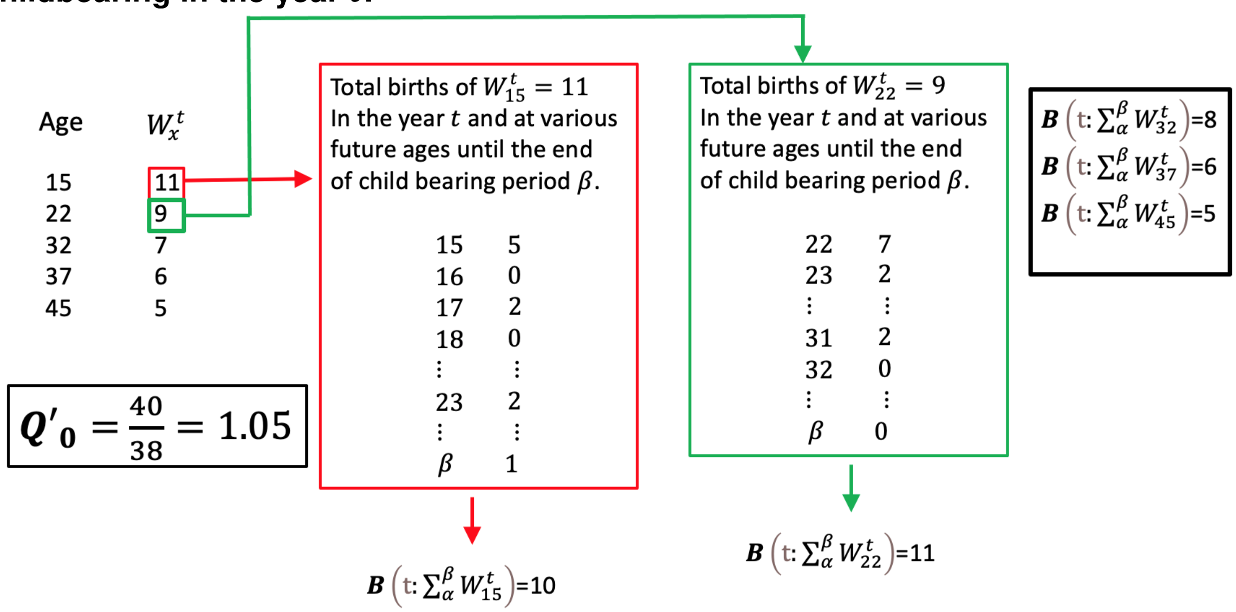

Figure 1. Computation of the average number of girl babies who will be replacing the current childbearing women. The total number of women in the age group W_x^t in the year t and their future births beyond the year t is used to compute replacement fertility. For the specific example it is assumed no mortality among 38 women until they finish childbearing, but that need not be the case. Suppose two women die before or after childbearing then would affect values at the numerator and denominator in the computation of $\boldsymbol{Q'_0}$.

There are several ways to address the uncertainty explained in this note. One could also obtain metrics such as $|Q_0 - R_0|$ under various stability conditions as in **[5].** All such discussions are out of the scope of the current technical opinion piece.

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