Proceedings of the 2014 Cincinnati-Dayton INFORMS Symposium

August 29, 2014
Wright State University, Dayton, Ohio

Institute for Operations Research and the Management Sciences

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Institute for Operations Research and the Management Sciences (INFORMS)
Cincinnati-Dayton INFORMS Chapter

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1. Introduction

Conference Committee Message

The organizing committee welcomes you to the Cincinnati-Dayton INFORMS Symposium. The symposium is hosted by Wright State University's Biomedical, Industrial, and Human Factors Engineering Department, with corporate sponsors: Perduco Group, Applied Research Solutions, and Goldmeier Consulting. We feel that the scientific and social exchange among symposium attendees will give us a much needed opportunity to interact in the Miami Valley area.

We have been pleased by the volume and breadth of submissions with topics ranging from X to Y and was more than sufficient to completely fill the entire day. The amount and variety of attendees has also been refreshing. We therefore hope that this will be the first of many technical symposiums offered by the Cincinnati-Dayton INFORMS chapter.

We hope that you will be able to participate in our social happy hour and attend as many talks as possible. If you need anything during your attendance, please do not hesitate to let us know.

-Cincy-Dayton INFORMS Conference Steering Committee

Conference Steering Committee

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Cincinnati-Dayton INFORMS Chapter and the Symposium

The Cincinnati-Dayton Chapter of INFORMS was established in 1995 as one of the regional chapters of INFORMS. Regional chapters are separate from student chapters and encourage interchanges between professionals, faculty, researchers, and students; regional chapters are relatively few in number and the Cincinnati-Dayton chapter is the only local regional INFORMS chapter in Ohio, Indiana, Kentucky or West Virginia. The Cincinnati-Dayton chapter has been successful in its mission by sponsoring the annual Arnoff lecture, offering chapter awards and social events, and encouraging facility tours, guest speakers and symposiums. Our chapter current has approximately 90 members, and 400 prior members, many no longer in the Miami Valley area. Your patronage of the symposium both helps us to expand our chapter and facilitate needed technical interchanges.

More details can be found on our chapter webpage:

https://www.informs.org/Community/Cincinnati-Dayton-Chapter

Presented Research and Keyword Index

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Our Master of Science features programs in the Biomedical Engineering and Industrial & Human Factors Engineering. Also, research is prominent in the Ph.D. in Engineering program in three focus areas: Industrial & Human Systems, Material and Nanotechnology, and Sensor and Signal Processing. The Master of Science in Industrial & Human Factors Engineering can be earned entirely online through Distance Education.

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Attendees and Presenters

The Cincinnati-Dayton INFORMS Chapter would like to thank the attendees and presenters, without whom this symposium would not be occurring. It is interesting and reassuring to see the diverse affiliations of those attending our symposium.

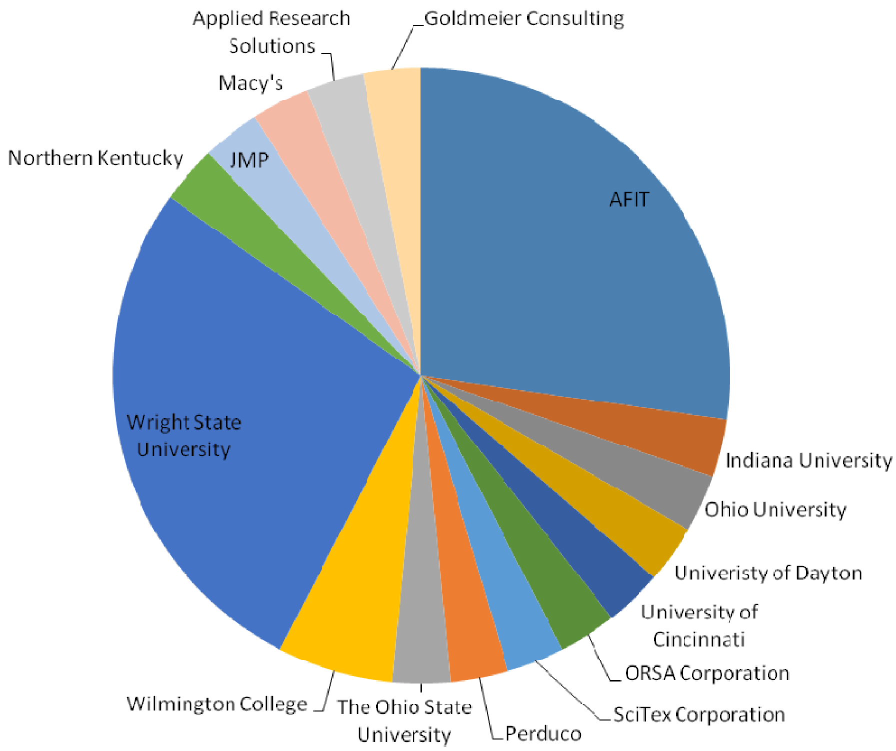
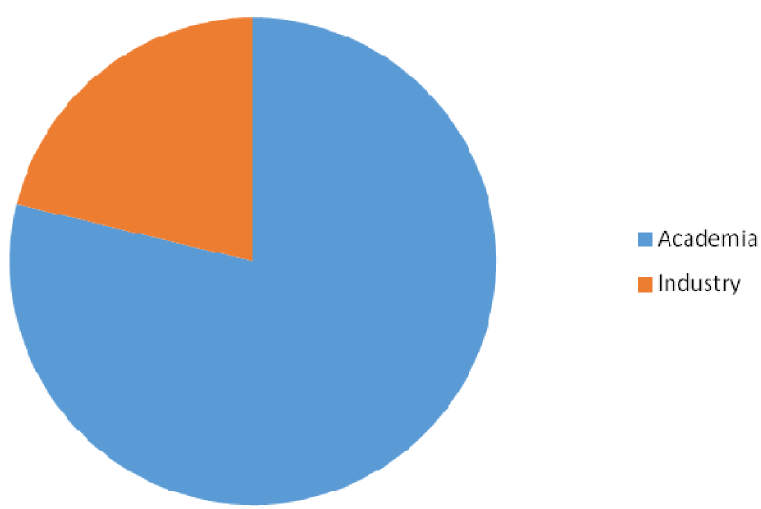


Figure 1, Presenter and Attendee Affiliations



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2. Schedule

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	Track 1 – Education and Decision Making ***Chair: James Cordeiro***	**Track 2 – Social Network Analysis**	**Track 3 –Poster Session** ***Chair:* Trevor Bihl**
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1320-1340			
1340-1400	***A Simulation of Decision-Making Under Imperfect Situation Analysis*** **Presenting author**: Victor Middleton, ORSA Corporation **Co-authors:** Frank Ciarallo, Wright State University **Abstract-8**		Student Posters
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	Track 1 – Sports Analytics	**Track 2 – Service and Healthcare** ***Chair: Kellie Schneider***	
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1500-1520		***Planning Inpatient Discharges at Hospitals*** **Presenting author**: Nicholas Ballester, Wright State University **Co-authors:** Pratik Parikh, Wright State University Nan Kong, Purdue University **Abstract-10**	
1520-1540		***Incentive-Compatible Multi-level Triage in Emergency Medical Services*** **Presenting author**: Eric Webb, Indiana University - Bloomington **Co-authors:** Alex Mills, Indiana University – Bloomington **Abstract-11**	
1540-1640	**Keynote address: "*Examining vaccine economics using operations research*"** **Presenting author**: Matthew JD Robbins, Air Force Institute of Technology		
1640-on	**Awards presentation, then Social and Networking**		

3. Keynote Address: Outstanding Young OR/MS Award Winner

<u>**Examining vaccine economics using operations research**</u>

Matthew JD Robbins (Air Force Institute of Technology)

Presentation

Abstract: *Vaccination is one of the most important and successful public health endeavors in human history, profoundly reducing mortalities caused by infectious diseases. In the United States, the incidence of many childhood diseases has dramatically decreased, even as the number of diseases preventable by vaccination has increased. The comprehensive success of large scale pediatric immunization programs results from the collaboration of an interdependent system of government and industry stakeholders. A stakeholder in this system acts independently in pursuit of its own interests; yet, the actions of one stakeholder may profoundly affect the welfare of another stakeholder. It is imperative that these stakeholders understand the nature of their interdependence and the holistic impact of their behavior on the entire vaccine market. Of particular concern is the economic competition within the vaccine industry, the impact of government regulatory policies on the vaccine industry, and the attendant impact on the vaccine system's ability to ensure the adequate provision of pediatric vaccines.*

This presentation highlights three research papers in which operations research methods are applied to aid market participants in making more informed decisions regarding the pricing and purchasing of vaccines in the public sector of the United States pediatric vaccine market. The first paper examines pricing strategies for pediatric combination vaccines and their impact on the United States pediatric vaccine market. The resulting analysis determines if a combination vaccine is competitively priced when compared to its competitors, for a given set of federal contract prices. The second paper presents a static Bertrand oligopoly pricing model that characterizes oligopolistic interaction between asymmetric firms in a multiple homogeneous product market. The repeated game version of the model enables examination of tacit collusion in an underlying market of interest. The third paper presents an operations research approach that addresses the issue of the pediatric vaccine industry's continuing viability from the perspective of the Centers for Disease Control and Prevention (CDC). The proposed model can be used to design a pricing and purchasing policy for the CDC that establishes a sustainable and stable capital investment environment in which the reliable provision of pediatric vaccines so essential to public health can occur.

Bio: *Matthew J. Robbins received the B.S. degree in computer engineering from the University of Arkansas, AR, USA, in 1999, the M.S. degree in operations research from the Air Force Institute of Technology, Wright-Patterson Air Force Base, OH, USA, in 2005 and the Ph.D. degree in industrial engineering from the University of Illinois, Champaign, IL, USA, in 2010. He is an active duty Air Force officer, serving an appointment as an Assistant Professor of Operations Research with the Department of Operational Sciences, Air Force Institute of Technology, Wright-Patterson Air Force Base, OH, USA. His research interests include sequential decision making under uncertainty, game theory, network science, and applications of operations research in the military and public health-care domains. Dr. Robbins has been recognized with a number of awards, most notably winning the 2011 Pritsker Doctoral Dissertation Award (First Place) from the Institute of Industrial Engineers (IIE).*

4. Invited Speakers

Abstract-I1
Berry 2
0920-1020

Utilizing Feature-Based Cost Estimation in Preliminary Design

Dale Masel (Ohio University)

Presentation

Abstract: *The cost to manufacture a product is determined by the product's design, so decisions made early in the design process can have a significant impact on the manufacturing cost. Therefore, to minimize the cost of manufacturing the product, designers should consider cost when they are evaluating the advantages of different designs.*

However, this is often difficult for design engineers to do. They often lack the expertise to determine the appropriate processes to use in manufacturing the product. In addition, even if the processes are known, the design may not be detailed enough to determine the time (and cost) to perform the process. Once the necessary details have been determined to a sufficient level to allow estimation of the time, it will often be too late to change the design even if it is found that the cost can be reduced.

One approach to dealing with these issues is the use of parametric cost estimation models. These models estimate cost as a function of just a few of the part's parameters, such as weight, overall length, or performance. However, parametric models provide limited insight on how to reduce the cost of a design, since only a reduction in one of the chosen parameters will reduce cost.

More useful information can be provided to designers by utilizing process-based models for the manufacturing processes that are performed. So that designers don't have to select the processes, the typical process plan for different part types can be predefined. And to deal with the limited design data available, the detailed process parameters can be calculated from the preliminary design information available, using relationships based on typical design practice.

This approach to cost estimation has been successfully implemented for a variety of products, including jet engines, gas turbines, and wind turbines. The approach has been shown to provide a high accuracy in estimating cost while also providing insight into how much different part features contribute to the part's cost.

Keyword: cost estimation

Abstract-I2
Berry 1
1020-1200

The Origins of Operations Research: Science at War

Michael W. Garrambone (InfoSciTex Corporation, a DCS Company)

Presentation

Abstract: *In 1934 Sir Henry T. Tizard, Rector of the Imperial College of Science and Technology was selected by the Air Ministry as chairman of the Committee for the Scientific Survey of Air Defence. It was from this appointment and the first meeting of the "Tizard Committee" that over the next five years the applications of Radar were explored by scientists and engineers who would become the first formal generation of what we know today as military operations research analysts. In hand with senior Armed Services officers such as Air Chief Marshal Hugh Dowding, the military drew closeness with their research and development scientists, and thus the serving officers and university research scientists aligned in confidence and mutual understanding to take on the eminent problems facing the defense of Britian. It was at his point, that the Services recognized that scientifically trained researchers could play a vital part not only in the development of weapons and tools of war, but also in the study and execution of military operations. Attendees will find this short presentation on the origins of Military Operations Research to be both enlightening and entertaining for it focuses on the early developments of our profession by our British counterparts—indeed, for their times and ours OR continues to be a revolutionary and important notion in making applications of science and scientists to the operational aspects of war.*

Keyword: military OR applications, history of OR

Abstract-I3
Berry 2
1300-1400

Introduction to Criminal Justice and Military Applications of Social Network Analysis

James Morris (Department of the Air Force)

Presentation

Abstract: *Social Network Analysis (SNA) employs a collection of methods to analyze the structure of human social networks. Primarily conducted to increase the efficiency of organizations or for researching aspects of collective human behavior, the past decade has found SNA incorporated into criminal justice and military operations. This introductory overview will discuss criminal justice and Department of Defense applications of SNA while providing a foundational understanding of rudimentary SNA techniques.*

Keyword: social network analysis, military OR application

Abstract-I4
Berry 1
1440-1540

<u>Sports Analytics</u>

Jacob Loeffelholz (The Perduco Group)

Presentation

Abstract: *Moneyball, Numbers Never Lie, Odds and Perduco Sports – all of these have one thing in common – Sports Analytics. Sports Analytics is a rapidly growing field in which more and more professional, collegiate, and amateur organizations are partaking. With the availability of massive amounts of data (whether good or bad), the need for analysts to organize, understand, and translate data into meaningful insight is imperative. Some of the top European football (soccer) clubs hire teams of PhDs to gain even the slightest edge over the competition.*

This talk will cover one small company's journey through the uncertain landscape of Sports Analytics. Our experience and engagement across the sports industry will be discussed, including applications related to professional and collegiate teams, fantasy sports, gaming, agent analytics and much more! A wide array of Operations Research techniques such as the Traveling Salesman Problem, Heuristics, Value Focused Thinking, Bayes Theorem and others will be demonstrated with regards to their application in Sports Analytics. Whether you enjoy your sport on turf, grass, hardwood or even ice, this presentation will cover it all!

Keyword: Sports, Hospitality, and Recreation

5. Abstracts for Presentations and Papers

Abstract-1
Berry 1
0920-0940

Multi-objective optimization of stochastic black-box systems using direct search and indifference values

Todd Paciencia (Air Force Institute of Technology)
James Chrissis (Air Force Institute of Technology)

Paper (pages: 28-53) and Presentation

Abstract: *In this work, a general framework is developed to solve black-box, multi-objective problems to a desired level of resolution or completeness of the Pareto front. This framework can be used to solve problems with or without closed form representation and can be expanded easily for stochastic responses. An indifference region-based method is developed to help determine the completeness of a Pareto approximation and to find any possible missing portions of the optimal front. This method is used with optimization of single-objective formulations via direct search methods to complete the approximation. The resulting algorithm is evaluated on systems with up to eight objectives and is shown to provide a reasonably complete approximation of the Pareto set, and to do so efficiently.*
Keyword: optimization, stochastics

Abstract-2
Berry 1
0940-1000

An Overview and Investigation of the Weapon-Target Assignment (WTA) Problem

Carl Parson (Air Force Institute of Technology)
Darryl Ahner (Air Force Institute of Technology)

Presentation

Abstract: *The weapon-target assignment (WTA) problem is a fundamental, and classic combinatorical non-linear optimization problem in the field of military operations research. The WTA can be found under many different names and formulations, all which share certain structural similarities. The generalized WTA problem will be presented, as will several of the variations found in literature. The structural components which can be exploited are discussed as well as some interesting results.*
Keyword: optimization, military OR applications

Abstract-3
Berry 1
1000-1020

The Optimal Synchronization of Average Throughput in Supply Chain Networks

Gregory Kellar (Wright State University)
George Polak (Wright State University)
Xinhui Zhang (Wright State University)

Presentation

Abstract: *We propose a mode of synchronizing discrete lots practicable for crossdocking via bulk-breaking or consolidation, indicated by the equality of average throughput at steady state.*

We formulate an original nonlinear mixed general integer program that determines optimal order lot-sizing that indicate the relative degree of "push" or "pull" between facilities.
Keywords: Logistics and Supply Chain Management, Linear/Nonlinear and Integer Programming

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Abstract-4
Berry 2
1020-1040

The Influence of Load on Service Times

Kenneth Schultz (Air Force Institute of Technology)

Presentation

Abstract: *Dependence of service times on load has been documented recently in various systems. We develop a general framework to help both empirical and analytical researchers to investigate and model how load impacts service times. We examine interactions among "load characteristics," "system components," and "service time determinants" while studying the effect of load on service times. We characterize load in terms of three dimensions: "changeover," "load," and "overwork." We distinguish between three system components: "server," "customer," and "network." We decompose service time into "work content" and the "service speed." To validate the framework, we use it to explain the results of published empirical papers that document dependency of service times on load. We illustrate use of the framework to generate hypotheses about service times in an EMS system.*
Keywords: Logistics and Supply Chain Management

Abstract-5
Berry 2
1040-1130

Intro to Data Visualization Principles

Jordan Goldmeier (Goldmeier Consulting Co LLC)

Presentation

Abstract: *Visualization is powerful tool to reveal insight and meaning within data. However, many organizations do not follow data visualization research when presenting information, creating environments for potentially misinformed and hazardous decision-making. In this workshop, I will present data visualization research and how best to use this information in your organization. Specifically, I will review Gestalt principles of perception and preattentive attributes, which describe how we perceive quantities in the visual world. Finally, we conclude with a few quality examples of good and bad data presentations.*
Keyword: Visualizations

Abstract-6
Berry 2
1130-1200

Improved Visualization of n-Dimensional Data Using Hyper-Radial Values

Todd Paciencia (Air Force Institute of Technology)
Trevor Bihl (Air Force Institute of Technology)
Kenneth Bauer (Air Force Institute of Technology)

Presentation

Abstract: *High-dimensional data is naturally difficult to visualize in a meaningful way, as anything with more than four dimensions provides challenges.[1] Unfortunately, many realworld*

datasets have much greater than four dimensions and have complex interactions between features, making a simple plotting of feature subsets impractical for most purposes. The problem of visualizing high-dimensional data has become increasingly relevant, as systems generate more data faster and processing efficiency demands continue to increase in order to find information. Decisions based on very large data has become critical to areas of business, clinical treatments, cyber and national security, and disaster management.[2] Visualization of data can help with parameter choice and understanding of data characteristics for many algorithms and applications. Additionally, visualization can provide confidence in data exploration and is more intuitive than complex algorithms.[3] For the purposes of this research, we are interested in being able to utilize visualization to identify general characteristics of a multivariate dataset such as overlap of classes and outliers. In the application of classification, this visualization enables data complexity comparisons, possible class identification, and an evaluation of linear or non-linear algorithm appropriateness. Many multi-dimensional visualization techniques exist, but frequently these are not intuitive or do not lend themselves to the visualization of many data features. Additionally, some become computationally expensive as the number of data features increases. Surveys of various methods include those by Chan [4], Kehrer and Hauser [5], Keim [3], Kromesch and Juhasz [6], and Grinstein, Trutschl, and Cvek.[7] Due to limitations of existing methods, the authors propose extending the Hyper-Radial Visualization (HRV) method for visualizing multivariate data.

The approach presented herein extends the HRV method of Chiu and Bloebaum for visualization of Pareto frontiers in multi-objective optimization problems.[8] This method is powerful in that data features are really only aggregated, vice transformed, to create the resulting visualization and generation of the visualization itself is very efficient. Whereas HRV was originally designed for comparison of competing optimal designs, we broaden its use for visualizing class and exemplar characteristics in multivariate data. In order to improve the visualization, we also present optimization strategies to generate the groups required for both supervised and unsupervised cases. Because as the number of features increases, any two-dimensional visualization becomes inherently limited in being able to display the information present, the authors also create a three-dimensional version to enable visualization for larger numbers of features. For examples, and in order to compare our HRV methods to existing visualization methods, we apply our visualizations to various well-known data. Clarification is necessary for the MNIST dataset, in that features (pixels) with zero range were removed. These sets were chosen to showcase flexibility to number of exemplars, number of features, number of classes, and general complexity.

After reviewing existing visualization methods next, we make the following contributions in order:

1) Extend HRV to multivariate data.

2) Develop optimal group algorithms for the HRV visualization, in both cases of having and not having class information.

3) Develop a three-dimension version of HRV incorporating our optimization strategies.

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Keyword: Visualizations, Data Mining and Applied Statistics

Abstract-7
Berry 1
1300-1340

Supplemental Instruction and Undergraduate Business Statistics Student Performance

Angela Mitchell (Wilmington College)
James Fitz-Simmons (Wilmington College)

Presentation

Abstract: *This presentation session is focused on Supplemental Instruction (SI) in business statistics courses. Often statistics classes can be challenging for undergraduate students. Supplemental Instruction sessions provide a means to facilitate learning in such challenging courses. SI is offered on a weekly basis during the semester. The SI sessions are one hour long and are led by a student who has previously excelled in the course. SI was developed in 1973 and is student-driven and "encourages collaborative learning." (A. Harding et. al, 2011). Several studies demonstrate that SI attendance improves course performance (A. Harding et. al, 2011; M. Oja, 2012; J. Price et. al, 2012). As SI is an optional part of our courses, we wanted to explore the correlations between SI participation and learning assessments in these courses to see if we should be further encouraging students to attend SI sessions or possibly to make attendance at them mandatory.*

Data were collected from two business statistics courses (Business Statistics I & II) at a small, private, liberal arts college. Business and accounting majors are required to take the course. Data collected included final grade, average quiz score, average exam score, and SI participation. We are using these initial data in a pilot study to explore correlations between SI participation and the learning assessment variables. This presentation will focus on the results of the analysis of the data collected as well as a discussion of where we might go from here.

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Keywords: Educational Practice, Data Mining and Applied Statistics

Abstract-8
Berry 1
1340-1400

A Simulation of Decision-Making Under Imperfect Situation Awareness

Victor Middleton (ORSA Corporation)
Frank Ciarallo (Wright State University)

Presentation

Abstract: *This presentation introduces a methodology to represent imperfect and uncertain situation awareness and situation understanding (SA/SU), as well as the effects of such SA/SU on decision-making. The methodology is implemented in an agent-based model (ABM) simulating a specific, easily understood, and quantifiable example of the impact of imperfect SA/SU on human behavior: intelligent agents being spatially "lost" while trying to navigate in a simulation world. The simulation is called MOdeling Being Intelligent and Lost (MOBIL).*

We present results of using MOBIL to investigate decision-making under uncertainty and error, by conducting a set of virtual experiments that examine how an intelligent agent's behavior is affected by information of varying levels and quality. These experiments vary aspects of an agent's perceived worldview to study how a mistaken understanding of ground truth affects achievement of the agent's goals. They provide insight into multiple aspects of decision-making as affected by problem complexity, information quality, risk tolerance, and decision strategies.
Keywords: Simulation (i.e. agent based & discrete event)

Abstract-9
Berry 2
1440-1500

Consideration of Product Exposure in Retail Design

Corinne Mowrey (Wright State University)
Pratik J. Parikh (Wright State University)

Presentation

Abstract: *A retail facility should effectively engage consumers during their shopping trips if they want to convert demand into purchases. Unfortunately, the complexity of the retailing environment and lack of scientific tools often results in gut-feel approaches experimented in practice. A key aspect of retail facility design, often alluded to but rarely analyzed, is product exposure to the shopper along their travel path. From a shopper's perspective, a greater amount of product exposure means less time spent searching for items of interest. From a manager's perspective, converting a shopper's time from searching to purchasing would likely result in increased sales. We define the extent of the shopper's field of vision in order to determine the actual exposure of products experienced by a traveling shopper. In so doing, we can explore the effect rack orientation has on product exposure. We also consider that some locations are exposed to traveling shoppers more frequently than others, referred to as the intensity of exposure, and explore how intensity changes with rack orientation. Our main contributions include defining product exposure and developing an approach to estimate it at any point along the travel path. Since changing the orientation of racks would also affect the overall space and shape of the sales floor, we develop a space model that is generalized for a variety of layouts. Our results indicate that certain rack orientations result in product exposures as high as 2.5 times that of the traditional 90° orientation.*
Keywords: Business Analytics

Abstract-10
Berry 2
1500-1520

Planning Inpatient Discharges at Hospitals

Nicholas Ballester (Wright State University)
Pratik J. Parikh (Wright State University)
Nan Kong (Purdue University)

Presentation

Abstract: *Recently, we completed a study with a local hospital to examine the day-of-discharge process and its effect on upstream patient boarding time. Using a simulation, we examined several strategies such as reducing discharge processing time and advancing discharge order writing earlier in the day. One of the examined strategies was actually implemented by the hospital and so far shows promising improvement. Now we ask the question, is there an optimal strategy that minimizes patient boarding time? If we assume that all discharge orders are written*

by 9 a.m., then the problem becomes that of finding a discharge schedule that minimizes boarding time of newly-admitted patients. We use data from a Trauma Unit at a Midwest US hospital for the year 2012 and consider a simulation-optimization approach to identify near-optimal sequences and compared them with traditional scheduling policies. A secondary measure of discharge lateness is also considered. Our results indicate that the discharge schedules are highly dependent on the nurse workload assignments and schedules of ancillary services.
Keywords: Health Care/Medical and Biomedical, Simulation (i.e. agent based & discrete event)

Abstract-11
Berry 2
1520-1540

Incentive-Compatible Multi-level Triage in Emergency Medical Services

Eric Webb (Indiana University, Bloomington)
Alex Mills (Indiana University, Bloomington)

Presentation

Abstract: *The Emergency Medical Services (EMS) system is designed to handle life-threatening emergencies, but a large and growing number of non-emergency patients are accessing hospital-based healthcare through EMS. A recent national survey estimated that 17% of ambulance trips to hospital Emergency Departments (EDs) were medically unnecessary, and that medically unnecessary trips make up an increasing proportion of all EMS trips. These non-emergency patients do not need the high level of care that an ED provides and could often be treated at an outpatient facility at considerably lower cost. Preliminary studies have shown that EMS ambulance workers could identify and filter out non-emergency patients with high accuracy, if given the chance. However, current reimbursement policies preclude this kind of triage at the ambulance, as most ambulance services only get reimbursed for providing transportation to the ED. Without triage at the ambulance, non-emergency patients often end up in congested ED waiting rooms for extended periods, because EDs strictly prioritize emergency patients. These non-emergency patients are therefore a prime target for reducing the load on EDs in order to meet quality-of-service goals, such as waiting time targets, without increasing costs or reducing quality of care. Our study uses a queueing model to examine the feasibility and benefit of constructing alternative arrangements whereby the hospital incentivizes the ambulance service to divert non-emergency patients to outpatient facilities. The usefulness of such arrangements depends upon the average hospital costs and reimbursements of emergency and non-emergency patients and on the current ambulance reimbursement rates through insurance and Medicare. We identify multiple scenarios where both the hospital and ambulance service can benefit by agreeing upon a certain level of triage by the ambulance service.*
Keyword: Health Care, Medical, and Biomedical

6. Abstracts for Posters

Abstract-P1
Berry 3

Using Past Scores and Regularization to Create a Winning NFL Betting Model

Eric Webb (Indiana University, Bloomington)
Wayne Winston (University of Houston)

Poster

Abstract: *Many papers have been published in recent decades discussing whether or not the National Football League (NFL) betting market is efficient. The authors have devised a betting model that would win 52.9% of the 7,554 bets against the spread it would have made over 33 NFL seasons, enough to make a profit. This performance is statistically greater than winning just 50% of the bets (p<.0001). Each week of the season, NFL scores from previous weeks are used to build a model estimating the point value of each team's offense and defense. These offensive and defensive point values combine with the average scoring in previous games and the average "home edge" of around 3 points to predict the scores for next week's games. These predictions are compared to an advertised point spread and a bet is made for the home/away team if the model predicts the home/away team will do better than the advertised spread suggests. The sum of the squares of the offensive and defensive point values are constrained to be less than a given regularization constant. Results from older weeks are discounted in the model via a weekly discount factor. The authors searched over all possible combinations of potential regularization constants and discount factors to find the combination that led to the best results. The bettor would win 52.9% of the games that do not push if he bets on every game. The bettor can be more selective of the games in which he bets and increase his performance further. For example, if the bettor only bets when the spread is 10 points or higher, he will win 54.6% of the 910 games that do not push.*

Keyword: Sports, Hospitality, and Recreation

Abstract-P2
Berry 3

Improving non-linear approaches to anomaly detection, class separation, & data visualization

Todd Paciencia (Air Force Institute of Technology)
Kenneth Bauer (Air Force Institute of Technology)
James Chrissis (Air Force Institute of Technology)
Mark Oxley (Air Force Institute of Technology)

Poster

Abstract: *In hyperspectral imagery (HSI), radiance is collected across hundreds of spectral bands for an area being imaged. As materials reflect electromagnetic (EM) energy differently, each pixel has a unique signature. We often seek pixels that are significantly different than the rest of the image in order to find objects of interest (anomalies).*

Linear methods have become popular for this problem and others due to their easy interpretation and speed. However, such methods can be improved when the data has non-linear

structure. Unfortunately, non-linear methods also increase computational requirements, and make interpretation more difficult.
Keyword: Visualizations, Image and Sensor Data Analysis

Abstract-P3
Berry 3

PHEV Battery Exchange Station Inventory Control Markov Decision Problem

Rebecca S. Widrick (Air Force Institute of Technology)
Sarah G. Nurre (Air Force Institute of Technology)
Matthew J. Robbins (Air Force Institute of Technology)

Poster

Abstract: *Increasing popularity of plug-in hybrid electric vehicles (PHEVs) has led to research in implementing battery exchange stations. These stations are similar to gas stations where a car can pull up and have their depleted battery exchanged for one fully charged. To manage an exchange station the number of batteries to charge at each time period needs to be determined in order to satisfy uncertain demand. Herein, we seek to determine the optimal charging policy for one exchange station over a 7 day period where exchange demand is distributed Poisson. We model this problem as a Markov decision process seeking to maximize net profit where the optimal policy is found using backward induction. This model is validated on a data set associated with an exchange station where many parameters are projections based on gas station usage.*
Keyword: Transportation

Multi-Objective Optimization of Stochastic, Black-Box Systems Using Direct Search and Indifference Values

Todd J. Paciencia[1] and James W. Chrissis[2]
Air Force Institute of Technology, WPAFB, OH, 45433

In this work, a general framework is developed to solve black-box, multi-objective problems to a desired level of resolution or completeness of the Pareto front. This framework can be used to solve problems with or without closed form representation and can be expanded easily for stochastic responses. An indifference region-based method is developed to help determine the completeness of a Pareto approximation and to find any possible missing portions of the optimal front. This method is used with optimization of single-objective formulations via direct search methods to complete the approximation. The resulting algorithm is evaluated on systems with up to eight objectives and is shown to provide a reasonably complete approximation of the Pareto set, and to do so efficiently if performed smartly.

Nomenclature

f^g = Utopia point

f^b = Nadir point

ω_i = Indifference value in objective i

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I. Introduction

THERE are many existing methodologies for optimization over multiple objectives, but most carry limitations. As discontinuity, non-convexity, a large number of objectives or decision variables, black-box systems, and/or uncertainty are introduced, existing methodologies may fail to find a relatively complete Pareto approximation in an efficient manner. During black-box, or simulation-based optimization this is very important, as each function evaluation may be expensive and decision-makers and analysts may not have enough knowledge of the system *apriori* to fix focus on only a part of the Pareto set. Therefore, the priority becomes achieving a desired level of completeness, while maintaining efficiency.

Three methods using direct search were introduced that each sought to avoid the limitations of other methods in order to solve a problem with no closed form representation relatively completely and efficiently. Walston introduced the first method, Stochastic Multi-Objective Mesh Adaptive Direct Search (SMOMADS).[1] SMOMADS uses aspiration and reservation levels within the context of scalarization functions to solve single-objective formulations. Using design of experiments with the aspiration and reservation level as factors, different regions of the Pareto front can be found. SMOMADS uses Ranking and Selection (R&S), a method where sample means are used to select a best candidate via probability of correct selection, to determine best points to account for variation.[2] The specific R&S procedure used by Walston was Sequential Selection with Memory (SSM) by Pichitlamken and Nelson.[3]

Audet, Savard, and Zghal introduced the second and third methods, Bi-Objective MADS (Bi-MADS) and Multi-Objective MADS (MultiMADS).[4,5] BiMADS is only for two objectives, and uses single-objective formulations and the ordering property in two objectives to complete the Pareto approximation. MultiMADS also uses single-objective formulations, but generates reference points for these formulations using an alternate simplex to the Convex Hull of Individual Minima (CHIM). Both of these methods were developed for deterministic problems, but are easily extended to the stochastic case by using R&S.

All of these methods use Mesh Adaptive Direct Search (MADS), or its mixed-variable form, MV-MADS, to solve their sub-problems.[6,7] MADS allows for an infinite set of directions to explore the set of decision variables and uses the concept of a mesh to control the search. More recently, OrthoMADS was introduced, where the polling directions are chosen deterministically such that they are orthogonal to each other so that the convex cones of missed directions at each iteration are minimal in size.[8] Further, the convergence results for OrthoMADS hold deterministically, rather than with probability one, and allow for non-linear constraints. The true limitation of using MADS is the growth of the poll set as the the number of decision variables increases.

This paper first discusses a few other multi-objective and black-box approaches for the sake of discussion. SMOMADS, BiMADS, and MultiMADS are then discussed in more detail. Next, an indifference-region based method is introduced to find potential missing parts of the Pareto front. This method is used to modify the MADS-based algorithms such that their objective functions can be used in a framework to create a n-dimensional algorithm, without the addition of further constraints or the need to build an alternate simplex. Results and parameter settings are also discussed.

II. Multi-Objective Optimization with a Stochastic Response

The multi-objective problem to be solved may include both continuous and discrete variables, as well as some level of noise or uncertainty in each objective. The specific problem formulation is:

$$\begin{aligned} \text{Minimize: } & E\left[F(x)\right] = E\left[f(x) + \epsilon_w(x)\right] \qquad (1) \\ \text{subject to: } & g_i(x) \leq 0, i \in \{1, \ldots, N\}, \\ & x \in \left\{\mathbb{R}^{n^c} \times \mathbb{Z}^{n^d}\right\} \end{aligned}$$

where $F(x) \in \left\{\mathbb{R}^{n^c} \times \mathbb{Z}^{n^d}\right\} \to \mathbb{R}^N$, $F = (F_1, F_2, \ldots, F_N)$ is the set of objective functions, $\varepsilon_w(x)$ is the random error or noise such that $E\left[\varepsilon_w(x)\right] = 0$, and x is the set of continuous and discrete design variables. Here, the objectives need not be smooth.

This formulation can have many optimal solutions depending upon the importance of the objectives to a decision-maker. Therefore, the Pareto set is the set of solutions such that no one solution is better than another in all objectives. Otherwise that solution is *dominated*. The deterministic form of this problem may be easier to solve, as points do not need to be sampled repeatedly to find a "best" response. Two points of significance in the multi-objective context are the *utopia* and *nadir* points. The utopia point, f^g, is the vector consisting of the minimum objective function value of each objective over all feasible points. This can be "easily" found by minimizing each objective independently. The nadir point, f^b, is the vector consisting of the maximum objective function value of each objective over all Pareto solutions. This is often approximated by the $pseudo-nadir$, found using the maximum objective function value of each objective over those solutions corresponding to f^g. This is an approximation, as there may be multiple optima for any objective.

A. Methods to Solve the General Problem

There are many multi-objective optimization methods, but those that can find a complete front for any system may still carry limitations once considering efficiency if applied to a stochastic response, black-box system. Messac and Mattson modified the Normal Constraint (NC) method to allow for exploration of the entire feasible space while finding an even distribution of points.[9] This method uses additional constraints built from the utopia plane to reduce the design space. For a stochastic, blackbox system, R&S may not be sufficient to deal with the uncertainty both in the objectives and these constraints. Shan and Wang[10] proposed a Pareto-Set-Pusuing (PSP) method to progressively sample closer to the Pareto front. This method is designed to generate even-distributed solutions, but cannot guarantee evenness or a complete front. Kim and de Weck designed an adaptive weighted sum technique to overcome the inability of weighted sum methods to find nonconvex solutions or to easily find evenly distributed solutions.[11] However, their technique uses additional constraints, user-defined parameters in addition to any needed for the sub-problem solver, and a complex construction of a Pareto front patching.

The use of surrogates is another tool often found in black-box optimization, and they can also be used various ways within the framework presented here if means or a best candidate are used for the responses. The use of surrogates is not a focus here, but is worthy of mention. It is important to note that metamodels are entirely reliant on proper sampling and in some cases, picking the correct parameters. Metamodels can have issues with highly non-linear or erratic systems, and discontinuous objectives. Mullur and Messac designed metamodels to find Pareto solutions after using NC to find near-Pareto solutions, but creating a non-linear metamodel for any of the sub-problems requires a certain number of near-Pareto solutions in an area.[12] Ryu, Kim, and Wan[13] used a meta-modeling, trust-region, and weighted sum combination to solve quadratic sub-problems to generate evenly distributed solutions in a bi-objective problem. However, they used a Central Composite Design to build the metamodels, again highlighting the possible issue of larger design spaces for surrogates. Jones, Schonlau, and Welch[14] designed an efficient, global optimization algorithm (EGO) using sampling, meta-modeling, cross-validation, and expected improvement. Couckuyt, et. al [15] developed an evolutionary algorithm to pick a best surrogate using expected improvement.

Although the research highlighted thus far is only a small fraction of the work related to multi-objective or simulation-based optimization, other non-heuristic methods may not generate a well-distributed Pareto front efficiently in the general case. It is important to note again, however, that using direct search to solve sub-problems for any algorithm may become inefficient as the number of decision variables increases. This will be discussed in further detail in the results. Next we will discuss the three MADS-based algorithms that we leverage in this work.

B. SMOMADS

SMOMADS solves Eq. (1) for Pareto optimal solutions by minimizing a single-objective formulation with a given aspiration level a and reservation level r,

$$S_a^r = -\left(\min(u) + \varepsilon \cdot \sum_{i=1}^{N} u_i\right) \tag{2}$$

where

$$u_i = \begin{cases} \alpha_i \cdot w_i \cdot (a_i - f_i) + 1, & f_i < a_i, \\ w_i \cdot (a_i - f_i) + 1, & a_i \le f_i \le r_i, \\ \beta_i \cdot w_i \cdot (r_i - f_i), & r_i < f_i, \end{cases}$$

$$w_i = \frac{1}{r_i - a_i},$$

$$\alpha_i = \begin{cases} (0.1)\left(\frac{r_i - a_i}{a_i - f_i^g}\right), & a_i \ne f_i^g, \\ (0.1)\left(\frac{r_i - a_i}{10^{-7}}\right), & \text{o.w.}, \end{cases}$$

$$\beta_i = \begin{cases} (-10)\left(\frac{r_i - a_i}{a_i - f_i^b}\right), & a_i \neq f_i^b, \\ (-10)\left(\frac{r_i - a_i}{10^{-7}}\right), & \text{o.w.,} \end{cases}$$

and where ε was set to 5 in Walston's work. The function u_i is of the type called component achievement functions, i.e. strictly monotone functions of the objectives. The minimization of Eq. (2) provides proper Pareto optimal solutions nearest the aspiration level. Therefore, sampling over a variety of aspiration and reservation levels can provide many solutions along the Pareto front. This is depicted in Figure 1. A deterministic dominance check is performed for SMOMADS. As a tolerance is not always easily defined, and the Pareto front may be unknown *apriori*, the deterministic check is also used for this research.

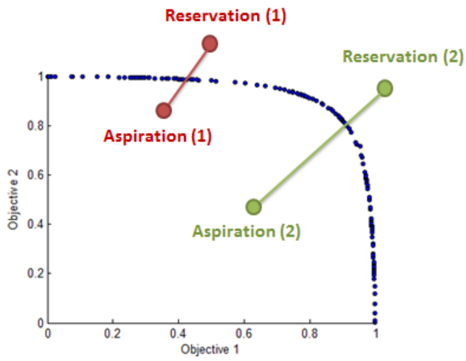


Fig. 1 Intersection of Rays with Pareto Front

C. BiMADS

BiMADS approximates two-objective Pareto fronts by solving a series of single-objective formulations.[4] Specifically, BiMADS begins by finding the solutions that correspond to the utopia point components. As evaluating these solutions for both objectives yields the pseudo-nadir, this also bounds the Pareto approximation. The algorithm works toward the Pareto front, using a weighting strategy such that each current nondominated point has a corresponding δ. This δ equals the sum of the distances from that point to its predecessor and successor in objective space (utilizing the ordering property), divided by the current weight. A point is selected using the maximum δ. A new single-objective formulation is then solved using a reference point derived from the maximum objective function values of those predecessor and successor points. The weights are adjusted so that no point is selected too many times, in the case of discontinuity. This creates a *gap*-filling strategy. Within the algorithm, every point evaluated is also considered for nondominance and the algorithm terminates once the maximum δ is below some predetermined value.

There are two specific single-objective formulations that may be used with a reference point r.

The single-objective normalized formulation is:

$$\hat{R}_r : \min_{x \in X} \left(\hat{\psi}_r\right) = \hat{\phi}_r \left(f_1(x), f_2(x), \ldots, f_N(x)\right) = \max_{i \in \{1,2,\ldots,N\}} \frac{f_i(x) - r_i}{s_i} \tag{3}$$

where $s \in \mathbb{R}^n$. The single-objective product formulation is:

$$\tilde{R}_r : \min_{x \in X} \left(\tilde{\psi}_r\right) = \tilde{\phi}_r \left(f_1(x), f_2(x), \ldots, f_N(x)\right) = -\prod_{i=1}^{N} \left(\left(r_i - f_i(x)\right)_+\right)^2 \tag{4}$$

where $(r_i - f_i(x))_+ = \max\{r_i - f_i(x), 0\}$ and $i = 1, 2, \ldots, N$. These formulations were shown to have convergence to Pareto solutions for any number of objectives using Clarke calculus for non-smooth functions.[4] The latter formulation restricts the choice of reference point whose dominance zone should be non-empty, but preserves differentiability of the original problem.[5]

D. MultiMADS

MultiMADS was designed by Audet, Savard, and Zghal[5] to solve problems with more than two objectives without restricting the choice of reference point. They proposed a new single-objective formulation:

$$\overline{R}_r : \min_{x \in X} \left(\overline{\psi}_r\right) = \overline{\phi}_r \left(f_1(x), f_2(x), \ldots, f_N(x)\right) = \begin{cases} -dist^2\left(\partial D, f(x)\right), & \text{if} f(x) \in D, \\ dist^2\left(\partial D, f(x)\right), & \text{o.w.}, \end{cases} \tag{5}$$

where $dist\left(\partial D, f(x)\right)$ is the distance in the objective space from $f(x)$ to the boundary ∂D of the dominance zone relative to r in the objective space. Here, the L_2-norm is used. The dominance zone D is defined as $\{x \in \mathbb{R}^n : f_i(x) \leq r_i \text{ for } i = 1, 2, \ldots, p\}$. They showed that this formulation provides a more flexible optimality condition than $\tilde{R}_r$ and that this formulation generalizes the $\hat{R}_r$ formulation. To construct a $y \in \partial D$ relative to r,

$$y_i = \begin{cases} r_i, & \text{if } i = \hat{i}, \\ f_i(x), & \text{o.w.}. \end{cases} \quad \text{for } i \in \{1, 2, \ldots, p\}, \tag{6}$$

where $\hat{i} \in argmin \left\{|f_i(x) - r_i| : i \in \{1, 2, \ldots, p\}\right\}.$

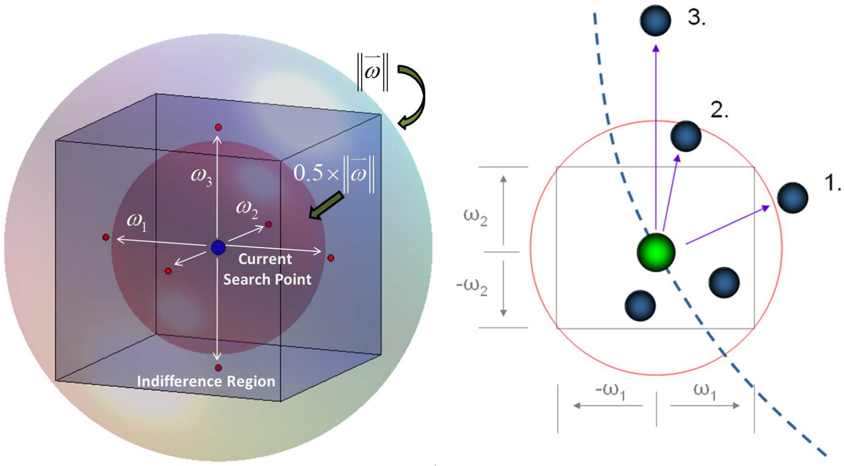
To determine the reference points, first $z^* = min_{x \in X} \sum_{i=1}^{p} s_i f_i(x)$ is found where s_i is a positive scaling factor. Then vectors are generated from the set $B = \{\beta \in \mathbb{R}^p : \sum_{i=1}^{p} \beta_i = 1, \beta_i \geq 0\}$. A reference point is defined as $r = f^g + z^* \beta I_p : \beta \in B$. The set of these reference points is referred to as the Tangent Hull, and is the alternate simplex. To generate a nice distribution of vectors to create the reference points, the strategy from Normal Boundary Intersection (NBI) is used.[16] In this work, scaling factors of 1 are used in the formulation.

III. Determining the Completeness of a Pareto Approximation

It has been shown for SMOMADS[19] that sampling the aspiration and reservation levels using space-filling designs such as Hammersley sequence sampling[17] and Near-Uniform Design (NUD)[18] can be a more reasonable approach to finding a complete front. Furthermore, MultiMADS has been shown to have desirable results on three-objective problems[5]. However, in some cases being able to apply more of a gap-filling approach such as that found in BiMADS may be a more efficient and/or more straightforward way to guarantee the generation of well-distributed Pareto approximations. As the single-objective formulations of all of these methods can still be used if we can determine reference points and gaps in the front, the true issue for greater than two objectives is how to determine if, and where, these portions of the Pareto front are missing from the approximation.

Wu and Azarm developed metrics to compare approximations[20] and Farhang-Mehr and Azarm developed an entropy metric to determine the true quality of the approximation of an unknown front.[21] A useful concept implemented by these metrics is an $indifference\ region$, or $indifference\ values$. Using indifference values, a decision-maker can attempt to *apriori* decide the required fidelity of the Pareto front appoximation in each objective, creating a hypercube or indifference region around each point. We will show that this can be used to find gaps with resepect to individual objectives in n-dimensional space.

Figure 2 depicts the basis behind Algorithm 1. In accordance with satisfying the decision-maker's preferences, each point on the Pareto approximation ideally has another point preceding and succeeding it within the indifference value ω_i (and region) for each objective i as appropriate. Such an indifference region is shown in Figure 2(a). By sorting data one objective at a time,



(a) (b)

Fig. 2 Searching Around a Point.

and searching along that objective starting from each data point, gaps may be found using the indifference values. However, as noted with BiMADS, sorting with more than two objectives no longer puts points into order such that successive points are necessarily closest to or are near each other in the space. Therefore, the distance between points relative to some fraction of a norm (here L_2 is used) of the indifference values can be used to ensure points under consideration are in the correct portion of objective space. This method helps account for the fact that solutions lie on a hypercurve, and are not necessarily "linearly" next to each other, and produces a gap consisting of two endpoints. A reference point for $\tilde{R}_r, \hat{R}_r, or \overline{R}_r$ can be constructed from these gap endpoints using their maximum values in the objective space. A reservation level for S_a^r can be constructed in the same manner, with the aspiration level being formed by the minimum values. We will show later that this very simple means of finding gaps can be used to form an effective multi-objective framework.

Figure 2(b) demonstrates a simple example. Assume the curve is the Pareto front, the green point is the current solution being searched around, and the square and circle denote the indifference hypercube and indifference L_2-norm respectively. When searching "above" in the second objective, Point 1 is within the indifference value, but outside the indifference norm. Point 2 is within the indifference value, but outside the indifference region. The algorithm would then evaluate Point 3, but as it and any succeeding points are outside the indifference value, a gap would be indentified.

Going by objective, this is a very quick way to search for gaps among multi-objective solutions. To rectify the issue of the L_2-norm not truly representing the indifference hypercube, 0.5 of the L_2-norm can be used, as shown in Figure 2(a).

Algorithm 1: Indifference Value-Based Gap Algorithm

Given $c > 0$, a vector of indifference values $\vec{\omega}$, and p non-dominated points:

1: Set $d_{crit} = c \cdot \parallel \vec{\omega} \parallel$.

2: Repeat for each objective n.

3:    Sort the objective data in ascending order of function value. Set $j = 1$.

4:       Repeat for each data point j, relative to the sorted data.

5:          Set $i = 1$.

6:          If $j = 1$ or $j = p$, set $j = j + 1$ or stop, respectively (extreme points).

7:          Else, if $|f_j^n - f_{j-i}^n| \leq \omega_n$ and $\parallel f_j^n - f_{j-i}^n \parallel \leq d_{crit}$, set $j = j + 1$.

8:          Else, if $|f_j^n - f_{j-i}^n| > \omega_n$, find the closest point k to j (smallest L_2-norm), from point 1 to $j - 1$.

9:             If $|f_j^n - f_k^n| \leq \omega_n$, set $j = j + 1$ (will find in another objective).

10:             Else, add (j, k) as a gap. Set $j = j + 1$.

11:             End If.

12:          Else, $i = i + 1$.

13:          End If.

14:       Search above using same process 5-12, except using $j + i$ instead of $j - i$ in lines 7-8, and points $j + 1$ to p in line 8.

15: Remove gaps with a distance between their centers less than d_{crit}, retaining one.

Algorithm 1 is the resulting algorithm. Keeping in mind efficiency, Algorithm 1 removes similar gaps in its final step. Gaps may be found that are near each other in space when there are many objectives, and a single sub-problem may fill gaps for multiple objectives. If it does not, the gaps are re-identified in subsequent steps of the main optimization algorithm. Other steps in the algorithm also serve to either reduce similar gaps found, or to increase computational efficiency.

Algorithm 1 does has one known limitation. A single point can satisfy both the "above" and "below" search for more than one objective. Imagine a circle of points on the surface of a sphere, depicted two-dimensionally in Figure 3. These points can satisfy indifference and distance criteria, while leaving a gap in the center of the circle. Fortunately, the iterative nature of the formulations and the mesh construct of MADS produces a low probability of such successive points re-occurring as the main algorithm progresses. Further, d_{crit} could be reduced, but this may affect the efficiency of the algorithm. In the event of such an occurrence, n-dimensional visualization techniques provide a means to potentially identify such gaps, after which reference points can be formulated from surrounding solutions to fill the gaps. Such techniques include Hyperspace Diagonal Counting (HSDC), where bins are created using a counting method away from the utopia to the nadir in two groups of objectives.[22] Parallel coordinates is another technique where objectives are shown on parallel y-axes, and the Hyper Radial Value (HRV) is a third technique that splits objectives into two groups and plots the hyper radius of each group from the utopia.[23]

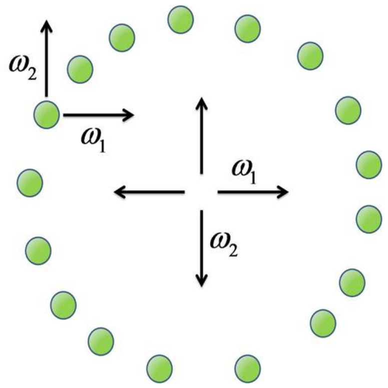


Fig. 3 Algorithm 1 Limitation.

IV. nMADS Algorithm

Adding Algorithm 1 and slightly changing the BiMADS strategy enables an n-objective algorithm, nMADS, shown as Algorithm 2. To use Algorithm 1 in this framework, decision-makers do not necessarily have to determine *apriori* their exact indifference values. Instead they may choose a number of bins in each objective and use the utopia and pseudo-nadir to derive values. The only caution in doing this is that the pseudo-nadir may vastly over-estimate the true nadir for certain systems, such as those with many localized fronts and multiple optima for the utopia. In these cases, adjusting the indifference values during the course of the algorithm is recommended, as otherwise convergence to the true optimal front will be slowed. In this respect, some *apriori* knowledge as to the scale of the objective space may be required. Indifference values set to approximately $\frac{f_i^b - f_i^g}{10}$ seem to work very well in practice.

Algorithm 2: nMADS

INITIALIZATION:

Let $size(g)$ denote the Euclidean distance between the two endpoints for a gap g.

1: Apply the OrthoMADS algorithm (with R&S if applicable) from initial iterate x_0 to solve $\min_{x \in X} f_i(x)$ for each objective $i = 1, \ldots, N$.

2: Remove dominated points and run Algorithm 1 to identify a set of gaps G, given some $c > 0$ and indifference value vector $\vec{\omega}$.

3: Initialize the weights $w(g)$ to $size(g)$ for all gaps $g \in G$. Initialize the weights $v(g)$ to 1 $\forall g \in G$.

MAIN ITERATIONS: Repeat while $G \neq \emptyset$ and $\max \{w(g)\} > c \cdot \parallel \vec{\omega} \parallel$

4: For each $g \in G$:

5: If $w(g) < c \cdot \parallel \vec{\omega} \parallel$, set $G = G \setminus g$, go to next gap.

6: Else:

7: Build reference point r by using maximum objective values from the endpoints of g.

8: Solve a single-objective formulation using the OrthoMADS (-R&S) algorithm from the starting iterate corresponding to one of the two endpoints of g.

9: End If.

10: End For.

11: Remove dominated points and run Algorithm 1 with resulting gaps G'.

12: If any center of $g' \in G'$ is within $\parallel \vec{\omega} \parallel$ of any center of $g \in G$ (according to Euclidean distance), set $v(g') = 2v(g)$, and set $w(g') = size(g')/v(g')$.

13: Else, set $w(g') = size(g')$ and $v(g') = 1$.

14: End If.

15: Set $G = G'$, REPEAT.

Using the same single-objective formulations from the algorithms in Section II, the reference points can still be built using the boundaries of an identified gap, as found by Algorithm 1. In the case of $\hat{R}_r$, $\overline{R}_r$, and S_a^r the optimal yields a solution in the dominance zone. Therefore, as Algorithm

2 progresses, gaps in the front are successively found and filled in to some acceptable resolution. Starting iterates are chosen from the gap boundaries with the intent of aiding direct search efficiency and because gaps are no longer tied to a "center" solution as with BiMADS. Weighting on the gap size is used to ensure that true discontinuities do not prevent termination of the algorithm.

Variations on the weighting scheme (such as an add-one to the denominator instead of doubling in Line 12), the starting iterate(s) to use for a sub-problem, a norm to use, and the sub-problem solver can also be applied to speed efficiency for a given problem. If using MADS, it is important to note that there is an optional search step where an experimental design can be used to sample on the mesh to try and speed convergence. Due to this and the pattern-based search, a single sub-problem may end up filling more than one gap or a gap with respect to mutliple objectives. Therefore, it may be even more efficient to only solve the sub-problem for the largest gap during an iteration. Additionally, other MADS parameters such as initial mesh size can affect the convergence rate. A function evaluation (FEval) limit is typically imposed to prevent too many calls to an expensive function while waiting for mesh convergence. Thus, it can be important to ensure that a high enough limit is used to find an accurate f^g or to allow the search to find improvement for a sub-problem. If there are localized fronts in the problem, due to the FEval limit imposed, it may be most efficient to first focus on those gaps closest to the utopia in hopes of screening out local Pareto solutions earlier. In the case when using R&S, the fact that solutions are being sampled repeatedly should also be a consideration for choice of FEval limits.

To exemplify Algorithm 2, we will first use the following three-objective problem, Viennet3:

$$\begin{aligned} \text{Minimize: } & F_1(x,y) = 0.5(x^2+y^2) + sin(x^2+y^2) \\ & F_2(x,y) = \frac{(3x-2y+4)^2}{8} + \frac{(x-y+1)^2}{27} + 15 \\ & F_3(x,y) = \frac{1}{x^2+y^2+1} - 1.1e^{-x^2-y^2} \\ \text{subject to } & -3 \le x, y \le 3 \end{aligned} \tag{7}$$

An initial approximation from searching for f^g is as shown in Figure 4(a). Algorithm 1 identifies gaps from the current approximation using $c = 0.5$ and indifference values chosen as 1/10 of the

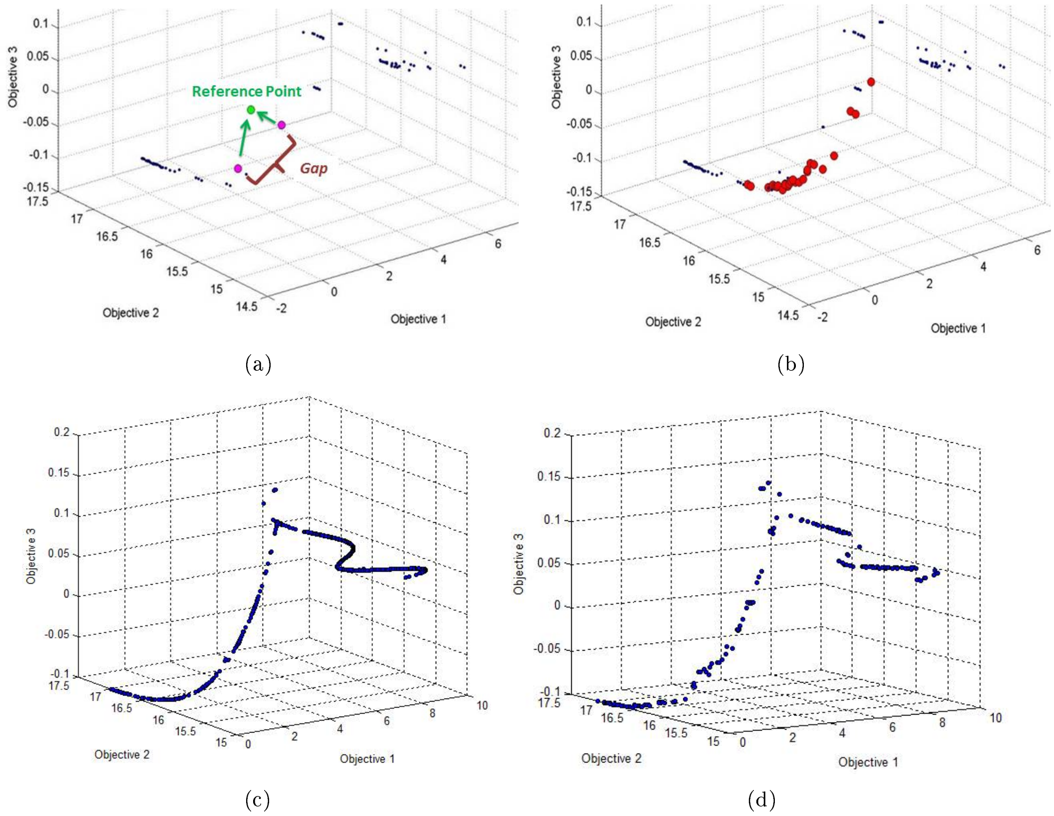


Fig. 4 Three-Objective Example.

difference between estimated utopia and pseudo-nadir components. In this case, one of the gaps identified was as shown, corresponding to Objective 3. Figure 4(b) depicts those points found using a sub-problem. Walston used SMOMADS to solve this problem, requiring 4,096 test points, or approximately 2,048,000 function evaluations to find a relatively complete front.[1] Instead, using Algorithm 2 with S_a^r, a 10-sample Latin Hypercube in the MADS search step, and a 500 FEval limit for all sub-problems, the front shown in Figure 4(c) was found. This only required 3,352 FEvals and found 1,163 unique Pareto solutions. Adding one percent of the pseudo-nadir objective values to each objective as noise, and using SSM as the R&S technique per Walston's work,[1] the front shown in Figure 4(d) was found. Although the effect of the noise and having to sample points repeatedly is evident, this is still relatively representative. This required 11,622 FEvals, and found 110 solutions.

V. Three to Eight Objective Problems using nMADS

In the remaining examples shown, default settings from Nomadm[7] were used, unless otherwise noted. For Algorithm 2, when the responses are treated as stochastic, noise is set to less than or equal to one percent of f_i^b, and f^g and f^b are estimated using two replications of the sub-problems. A 10-sample Latin Hypercube is used in the MADS search step, and the zero vector is used for x_0. No cache of function evaluations was maintained between sub-problems. These settings are meant to create a general aplication of the algorithm and to showcase its completeness, and are not necessarily good or optimal for a problem. The number of function evaluations and unique solutions are used as a comparison. Although the framework still works in only two objectives, it would be very similar to BiMADS and so results are only given for more than two objectives.

A. 3 Objectives

Having seen Viennet3, we will next look at a problem with many local fronts, and then one with disconnected Pareto regions. Consider the test problem DTLZ3:[24]

$$
\begin{aligned}
\text{Minimize: } & F_1(X) = (1+g(X))cos(x_1\pi/2)cos(x_2\pi/2) \\
& F_2(X) = (1+g(X))cos(x_1\pi/2)sin(x_2\pi/2) \\
& F_3(X) = (1+g(X))sin(x_1\pi/2) \\
\text{subject to } & 0 \leq x_i \leq 1 \\
\text{where} g(X) = & \; 100\left[|X| - 2 + \sum_{i=3}^{|X|}(x_i - 0.5)^2 - cos\left(20\pi\left(x_i - 0.5\right)\right)\right].
\end{aligned}
\tag{8}
$$

This problem has $3^{|X|-2} - 1$ local Pareto-optimal fronts, and a pseudo-nadir can highly overestimate the nadir. The global optimal front lies on the unit sphere. Figure 5(a) shows the front found for the four-variable problem, using Algorithm 2 with $\overline{R_r}$, an initial mesh size of 0.1, a FEval limit of 500 for all sub-problems, and $\omega_i = 0.1$. Just over 1,890 solutions were found in 6,500 FEvals. When the initial mesh was changed to 1, just over 2,200 solutions were found in 13,200 FEvals. Figure 5(b) depicts the front found again using an initial mesh of 1, but with using the solution found closest to the utopia as a starting iterate. This found almost 2,150 solutions in 12,900 FEvals. It is clear that various parameter settings may impact the efficiency of the algorithm on more difficult problems, but it is also evident that the algorithm consistently finds a relatively even and complete front.

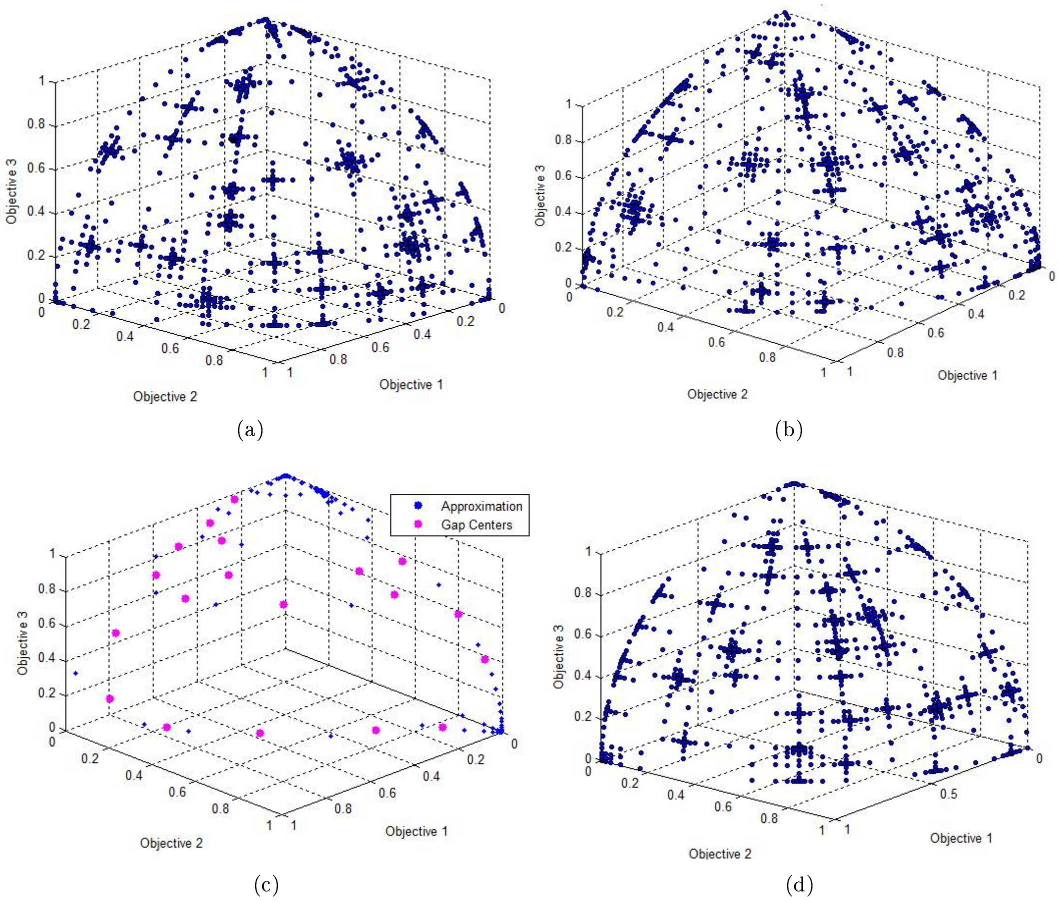
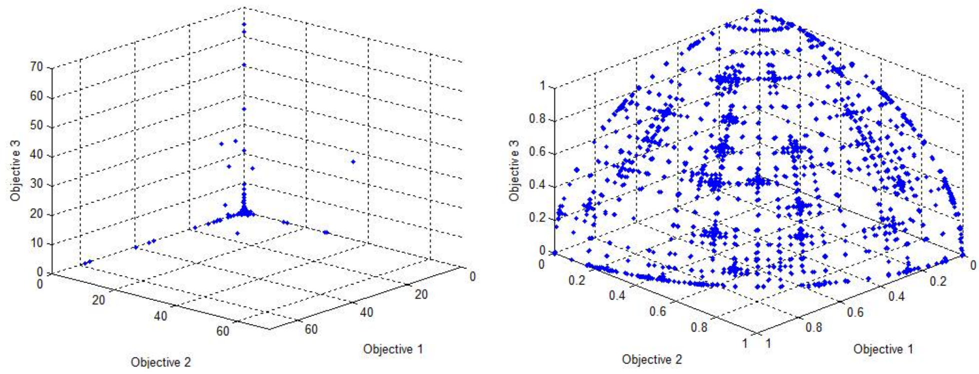


Fig. 5 DTLZ3 Deterministic.

Next, the number of variables was changed to 12 and the FEval limit to 1,000 for a sub-problem. Figure 5(c) is included to further clarify the notion of the algorithm. After the first iteration of gap-filling based on the search for the utopia, the approximation was as shown. The magenta points depict the means of the gap endpoints for each identified gap. Continuing the algorithm, the front in Figure 5(d) was obtained, consisting of 1,759 solutions and requiring a total of 24,067 FEvals. For a more complete front, with the trade-off of efficiency, a lower weighting scheme could be used. Here, there were $3^{10} - 1$ local fronts. Performance on this problem can be variable in that if the FEval limit is not chosen well, points found on these local fronts create gaps that may persist and thus unnecessarily increase total FEvals used until a point is found that dominates them. In this sense, an additional filter could be useful to screen for such points and either remove them or increase their associated sub-problem's FEval limit.

Next, we consider the stochastic response case for the four-variable problem, again using 500



(a) (b)

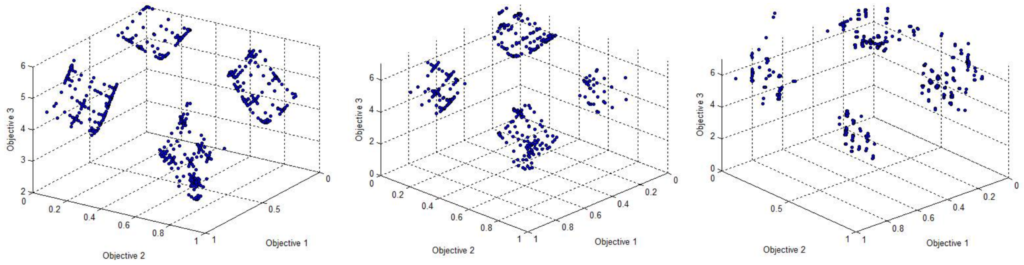
Fig. 6 DTLZ3 Stochastic.

as a FEval limit. Figure 6(a) shows an approximation after 75,000 FEvals. We see that the noise in the objectives has allowed bad solutions to escape the deterministic dominance check. However, if we ignore these points and scale to the desired front (as seen in Figure 6(b)), we have in fact obtained a good approximation. In fact, this front had nearly 1,350 solutions.

Now consider DTLZ7:[24]

$$\begin{aligned} \text{Minimize: } & F_1(X) = x_1 \\ & F_2(X) = x_2 \\ & F_3(X) = (1 + g(X))h(F, g) \\ \text{subject to } & 0 \le x_i \le 1 \\ \text{where} & g(X) = 1 + \frac{9}{|X| - 2} \sum_{i=3}^{|X|} x_i \\ & h(F, g) = 3 - \sum_{i=1}^{2} \left[\frac{F_i}{1+g} \left(1 + sin\left(3\pi F_i\right)\right)\right]. \end{aligned} \tag{9}$$

Again using $\overline{R_r}$, FEval limits of 1500, $\vec{\omega} = [0.1, 0.1, 0.2]$, and 22 variables, the front shown in Figure 7(a) was found. This took 20,950 FEvals with 1,369 solutions. However, as Pareto-optimal solutions have $x_{3,\ldots,20} = 0$, a starting iterate with $X = \vec{0.5}$ is more difficult. Adjusting the initial mesh size to be 0.1 so as to be smarter relative to the variable range, Figure 7(b) shows the front found using this new starting iterate. This found 699 solutions in just over 67,000 FEvals. Now using a stochastic response with the $X = \vec{0.5}$ starting iterate, the front shown in Figure 7(c) was



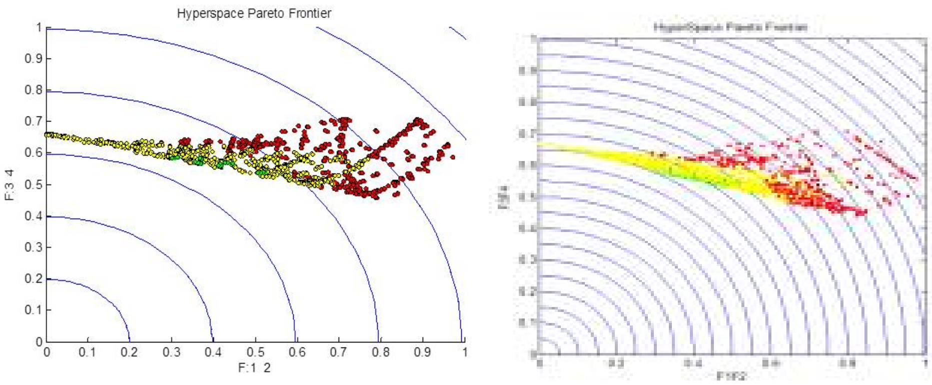
(a) (b) (c)

Fig. 7 DTLZ7.

found. This found 470 solutions but required 527,812 FEvals. Part of this was due to the trade-off between R&S and being able to move towards a better solution during a sub-problem. Additionally, noise was added to F_3 in addition to that already present in F_1 and F_2, perhaps making the problem even more difficult. It is important to note here that these results are still fairly efficient given the number of decision variables and the nature of direct search.

B. More Than 3 Objectives

Using the HRV representation, that groups normalized objectives into two groups and plots their hyper-radial values, we can now also showcase the benefit of Algorithm 2 using problems in more than three objectives. We use a few problems that have deterministic published solutions using HRV. The coloring of solutions depicted are HRV schemes to represent if all objective function values for that solution are within some value of the utopia,[23] and is not important here. We will use $\hat{R}_r$ as we have yet to show results using that formulation. All following results used stochastic responses, default Nomadm parameters, both gap endpoints as starting iterates (replicated sub-problems), a "plus-one" weighting strategy for the gap denominators instead of doubling, and indifference values derived from the estimate of $\frac{f_i^b - f_i^g}{10}$.



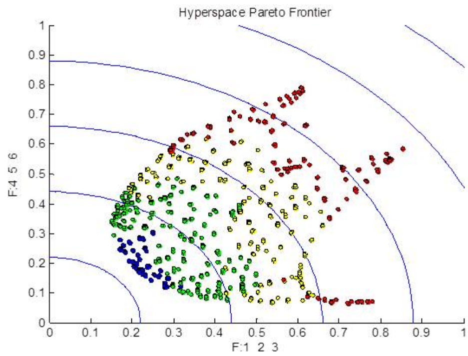
(a) (b)

Fig. 8 4-Objective HRV Pareto Representation.

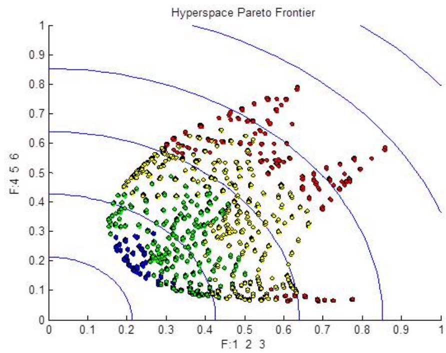
First, consider the following four-objective problem:

$$
\begin{aligned}
\text{Minimize: } & F_1(X) = 7.49 - 0.44x_1 + 1.16x_2 - 0.61x_3 \\
& F_2(X) = 4.13 - 0.92x_1 + 0.16x_2 - 0.43x_3 \\
& F_3(X) = -21.9 + 1.94x_1 + 0.3x_2 + 1.04x_3 \\
& F_4(X) = 11.33 - x_1 - x_2 - x_3 \\
\text{subject to } & F_1(X) - 7.49 \leq -3.1725 \\
& F_2(X) - 4.13 \leq -8.042 \\
& 1.94x_1 - 0.3x_2 - 1.04x_3 \leq 18.4988 \\
& 6.3969 \leq x_1 \leq 7.0901 \\
& 0.6931 \leq x_2 \leq 2.8904 \\
& 3.912 \leq x_3 \leq 4.6052
\end{aligned} \tag{10}
$$

Figure 8(a) shows a result, where $[7, 2, 4.5]$ was used as the starting iterate. A FEval limit of 500 was used to find the utopia, and 150 was used for the gap sub-problems. Here, 1,414 solutions were found in 5,176 function evaluations. The published solution is shown in Figure 9(b).[25] Although the proposed framework is somewhat robust to parameter settings in finding a complete front, as has been shown, parameter settings can have impact on the efficiency of the approximation. Here, high FEval limits were not needed and so sub-problems could be replicated and gaps weighted less



(a)



(b)

Fig. 9 6-Objective Solutions.

so as to help mitigate the effect of noise.

Consider the following six-objective problem:

$$
\begin{aligned}
\text{Minimize: } & F_1(x_1,x_2) = x_1^2 + (x_2-1)^2 \qquad (11)\\
& F_2(x_1,x_2) = x_1^2 + (x_2+1)^2 + 1\\
& F_3(x_1,x_2) = (x_1-1)^2 + x_2^2 + 2\\
& F_4(x_1,x_2) = \frac{(x_1-2)^2}{2} + \frac{(x_2+1)^2}{13} + 3\\
& F_5(x_1,x_2) = \frac{(x_1+x_2-3)^2}{36} + \frac{(-x_1+x_2+2)^2}{8} - 17\\
& F_6(x_1,x_2) = \frac{(x_1+2x_2-1)^2}{175} + \frac{(-x1+2x_2)^2}{17} - 13\\
\text{subject to } & -2 \le x_1, x_2 \le 2
\end{aligned}
$$

Figure 9(a) depicts an approximation using nMADS, a FEval limit of 500 for the utopia, and a limit of 150 on the sub-problems. To investigate variability of efficiency, 20 runs were conducted of the double-weighting scheme against adding one to the weight denominator each iteration. An average of 50 more solutions were found using the latter "plus-one" scheme. However, an average of 400 more function evaluations were required. Table 1 shows metrics for the 20 double-weighted scheme runs. Figure 9(b) depicts an approximation using the "plus-one" weighting scheme and the same 150 function evaluation limit, with an additional iteration of gaps filled once the algorithm had terminated. In total, over 17,000 function evaluations were used to find 3,333 solutions. This demonstrates that in latter stages of Algorithm 2, or with non-optimal points that have yet to be dominated, failing to adjust parameter values based on observation may cause unnecessary expense.

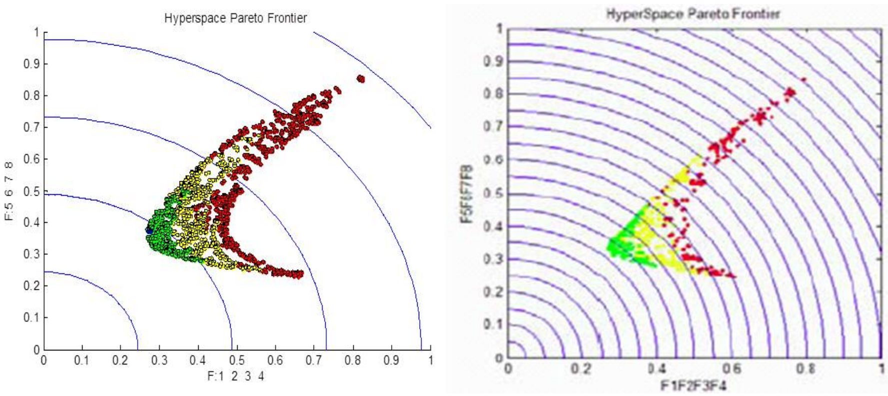
Table 1: 6-Objective Problem Metrics

	FEvals	Solutions
Mean	8897	1798
St Dev	638	127
Max	9894	2013
Min	7819	1553

Now we consider the following eight-objective problem to show the ability to find a relatively complete and even front in a large number of objectives:

$$
\begin{aligned}
\text{Minimize: } F_1(x,y) &= \frac{(x-2)^2}{2} + \frac{(y+1)^2}{13} + 3 \\
F_2(x,y) &= \frac{(x+y-3)^2}{175} + \frac{(2y-x)^2}{17} - 13 \\
F_3(x,y) &= \frac{(3x-2y+4)^2}{8} + \frac{(x-y+1)^2}{27} + 15 \\
F_4(x,y) &= \frac{(3x+y+9)^2}{34} + \frac{(x+1)^2}{15} + 29 \\
F_5(x,y) &= \frac{(4x-y-4)^2}{22} - \frac{(y-1)^2}{5} - 17 \\
F_6(x,y) &= \frac{(y+14)^2}{8} + \frac{(x+y)^2}{10} + 64 \\
F_7(x,y) &= \frac{(17-x-y)^3}{995} + \frac{(8y-5x)}{65} \\
F_8(x,y) &= \frac{(7+2x+5y)}{5} + \frac{(y-3x)^3}{235} \\
\text{subject to } & 4x+y-4 \le 0 \\
& -1-x \le 0 \\
& x-y-2 \le 0 \\
& -4 \le x, y \le 4
\end{aligned} \tag{12}
$$

Figure 10(a) shows the nMADS result in comparison to the published solution, Figure 10(b), again using HRV to plot the data. The published solution was found via a genetic algorithm and had 625 solutions.[23] nMADS (Algorithm 2) used 6,992 function evaluations and found 2,350 solutions



(a) (b)

Fig. 10 8-Objective Solutions.

in this instance, using the "plus-one" weighting scheme and a function evaluation limit of 150 for each sub-problem. Over 20 replications of nMADS without any additional iterations, an average of 1,967 solutions were found using 5,773 function evaluations. The respective standard deviations were 459 for function evaluations, and 154 for the number of solutions.

VI. Conclusion

In the stochastic case where R&S is required, the number of function evaluations may still be too high for some real-world problems due to expense of a single function evaluation. Further, as the number of decision variables or local Pareto fronts increases, the number of evaluations needed for a sub-problem is likely to increase. There are several options to increase efficiency that warrant more rigorous investigation; these include: adaptive indifference regions, adaptive mesh parameters, lowering the number of evaluations by R&S, choosing a best formulation for a problem, using different norms in the algorithms, and the use of surrogates. The algorithm could also be made more efficient by development of a filter for clearly non-optimal solutions that persist until that correct portion of the global front is found. Otherwise such points that occur on more difficult or noisy problems may cause the need for a larger FEval limit or repeated use of a sub-problem, and may take several iterations to progressively work towards the global front.

Using the new iterate strategy and gap location algorithm enable the driving concepts behind BiMADS to work in more than two objectives. Further, it seems that if used smartly, this framework allows for a robust, yet efficient, approximation relative to any multi-objective problem. The results on problems tested with up to eight objectives are promising, and have been similar on a variety of other problems not shown here. nMADS' efficiency relative to other algorithms wanes as the number of decision variables increases due to direct search. However, nMADS avoids limitations of many other multi-objective optimization algorithms and enables the solution of large numbers of objectives. Unfortunately, parameter settings are not always trivial to create a most efficient instance for a problem, and in the stochastic case, a large number of function evaluations may be unavoidable. These aspects require further study.

VII. Acknowledgements

I must give a special thanks to Dr. James Chrissis for his continued support, assistance, and patience.

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