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Contents

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Chair's Column

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It is heartening to see that despite its long history, optimization as an area is continuing to grow methodologically as well as in its applications. Whether it is saving lives, management of natural energy sources, designing better materials, improving return on investments, or machine learning with large datasets - optimization applications are everywhere - fueling the need to grow the field towards developing more realistic models, as well as faster algorithms and their implementations.

The present issue of the INFORMS Optimization Society newsletter, "INFORMS OS Today," features articles by the 2012 OS prize winners: András Prékopa (Khachiyan Prize for Lifetime Accomplishments in Optimization), Michel X. Goemans (Farkas Prize for Mid-career Researchers), Sergei Chubanov (Prize for Young Researchers), and Diego A. Morán R. (Student Paper Prize). These articles summarize their motivation of working on optimization problems, and the prize winning works. In his article "Bridging the Gap between Theory and Applications," András Prékopa takes us through his life journey and key problems motivating his work in stochastic and probabilistic constrained optimization. In a similar spirit Michel Goemans's article "My Path through Combinatorial Optimization" takes us through his work in combinatorial optimization, approximation algorithms and the max-cut problem. Sergei Chubanov summarizes his very interesting work on linear programming, presenting new polynomial complexity algorithms motivated from Pythagorean theorem and the relaxation methods of Agmon, and Motzkin and Schoenberg. Finally, the article by Diego Morán describes his strong duality results for Conic Mixed-Integer Programs.

In this issue, we also have announcements of key OS activities: calls for nominations for the 2013 OS prizes, a call for nominations of candidates for OS officers, and a call for hosting the OS 2014 Conference. The previous four conferences have had a diverse set of themes: "Optimization and HealthCare" (San Antonio, 2006); "Theory, Computation and Emerging Applications" (Atlanta, 2008); "Energy, Sustainability and Climate Change," (Gainesville, 2012); and "Optimization and Analytics" (Coral Gables, 2012). Please consider being active in the nomination process, as well as hosting the 2014 OS conference.

Optimization Society has traditionally had a very strong organized presence at the annual INFORMS meetings, the next one being at the Minneapolis Convention Center and Hilton Minneapolis on October 6-9, 2013. Our participation is organized via the OS sponsored clusters, which are organized by our Vice Chairs:

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Our very strong presence within INFORMS is due to the hard work of our vice-chairs, and it reflects the very large membership of the OS. Please contact appropriate Vice Chairs to get involved. I want to remind you that Pietro Belotti continues to be the OS webmaster, and he is always pleased to get your feedback on our website: `www.informs.org/Community/Optimization-Society`.

I look forward to seeing you at Minneapolis in October, in particular, at the OS Prize Session, and the OS Business Meeting. The latter is always one of the highlights of an INFORMS meeting to have some refreshments, meet with old friends and make new ones.

Bridging the Gap between Theory and Applications

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First I would like to express my thanks to the Committee: Kurt Anstreicher (Chair), Egon Balas, Claude Lemaréchal, Éva Tardos, for awarding me the 2012 INFORMS Optimization Society Khachiyan Prize. This paper summarizes my most important results in optimization.

I was born in Nyíregyháza, Hungary, and graduated from high school in the same town. After graduation in 1947 I joined the nearby University of Debrecen. During my university years a young professor of mathematics was appointed: Alfréd Rényi. He brought fresh air, introduced us to modern probability theory and emphasized the importance of applications. He impressed me a great deal and after he became the director of the Institute for Applied Mathematics of the Hungarian Academy of Sciences (H.A.S.) in Budapest, he took me there and I became his Ph.D. student. That status was called aspirant and meant independent study with the supervisor for three years. The degree received after the defense of a thesis was called candidate of mathematical (or other) sciences. I received it in 1956 from L. Eötvös University of Budapest. Ph.D. did not exist for a few years but it was reestablished after 1956, and those who became candidates earlier, received Ph.D. automatically. I received it from the University of Budapest, in 1960. My thesis was about stochastic set functions, random measures in abstract spaces. I proved extension theorems from algebra to σ-algebra and defined Poisson, Gaussian and more general random measures.

1. OR in Hungary and my life

I received the prestigious Grünwald prize for my thesis in 1956 and after the defense became assistant professor at the L. Eötvös University of Budapest. I kept my connection to the (at that time already renamed) Mathematical Institute of the H.A.S., where, in 1957, I started a seminar on Operations Research. The book by Charnes, Cooper, Henderson, Linear Programming [4] could be found in the library of the Institute. I was fascinated by the possibility to describe industrial and other social activities by convex polyhedra in multidimensional spaces and solve extremum problems by exact finite methods. I spent a lot of time making the subject attractive for students in mathematics and in the academic year 1958-59 gave my first course on Game Theory and Linear Programming that I repeated in the subsequent years. A few years later I published a book on the subject [13].

In 1959 I founded the first scientific research group on OR in Hungary at the Mathematical Institute of the H.A.S. under the name: Applications of Mathematics to Economics and became the head of it in part time. One of my successes was the organization of the 1963 international conference in Budapest entitled: Applications of Mathematics in Economics.

There is a Hungarian saying: if somebody says A, he has to say B, too. For operations research, for me it meant: if somebody decided to do it, he has to accept what that science is all about and do all of its activities: mathematics, applications, computing. The problem in Hungary was, however, that it was the mathematical life, where I could hope to realize my dreams but the mathematicians were not receptive enough. It is a long story how I did it, with my students and colleagues, but my school became the leading OR force in Hungary and its members were and are primarily mathematicians. I created an OR master curriculum at the University of Budapest in 1968, a Committee on OR in the Department of Mathematics of the H.A.S, organized several international and domestic conferences, and supervised a large number of Ph.D. students. The number of those who successfully defended thesis under my supervision is 54, as of today, and 5 are currently working. Among those are 16 RUTCOR students.

In Hungary I was teaching at the L. Eötvös University of Budapest (1956-1968, 1983-1985) and the Technical University of Budapest (1968-1983) and had part time jobs at the Mathematical Institute of the H.A.S. (1959-1970), the Computing Center and the Computing and Automation Institute of the H.A.S (1970-1985). In 1985 I accepted the invitation to join RUTCOR, and have spent almost twenty eight fruitful years there, the longest time I have spent at any university.

During this time I published my book on Stochastic Programming [23] and received important distinctions: Széchenyi Prize from the Hungarian Government (1996), Gold Medal of the European Operational Research Societies (2003), Khachiyan Prize (2012).

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2. The Hungarian Inventory control model

In 1962 the president of the National Planning Board asked my OR Group to look at the inventory control problems of the Hungarian enterprises and factories. They followed a rule to proportionally increase the inventories with the increase of production. Huge inventories accumulated nationwide and our job was to introduce new, more practical rules.

I personally took the job and worked on it with my colleague Margit Ziermann and two outside economists. The main problem was the randomness in delivery times and amounts of basic materials and semi-final products: the contracts between parties fixed the times and amounts for periods, e.g., quarters of years and the material supply departments tended to balance randomness by large initial safety stocks. There were no models suitable for applications in the already existing inventory control literature. The problem could be formulated (separately for each material) as: find the smallest initial inventory level that ensures continuous utilization by a probability at least $1-\epsilon$, where ϵ is prescribed and small. After several months of study the situation we started to apply order statistical formulas (for a summary, see [24]) that I subsequently generalized using deep measure theoretical tools. At that time we always came up with formulas for the initial inventory, corresponding to prescribed safety level.

The models were introduced nationwide and those were practically the only inventory control models applied in Hungary for more than twenty-five years. Right at the introduction of the models the savings amounted to an equivalent of today's four billion dollars. The overall savings in the course of the subsequent years cannot be assessed but our solution of the real life problem created recognition for operations research.

Later on several related optimization models have been formulated, suitable for application in cases of more general delivery and utilization processes. The first one in this respect was Prékopa, Kelle [27].

3. Programming under probabilistic constraint

Charnes, Cooper, Symonds [5] published a paper, where they solved an oil heating problem and introduced the idea: to impose a lower bound on the probability that a stochastic constraint should hold. Subsequently, Charnes, Cooper published other papers in which they formulated their decision principle and came up with some theory. Stochastic programming started by the works of Dantzig [6], Beale [1] and the mentioned paper [5]. Coming to OR from probability theory, I was dissatisfied with the formulation in [5]. The probability bounds were used individually for each stochastic constraint, neglecting the stochastic dependence between the random variables. Sometimes the individual "chance constraints" were even meaningless as I demonstrated it in network examples, where stochastic constraints jointly ensure reliability of the system. I formulated the problem with joint probabilistic constraint, as I called it. My first report on it was presented at the Princeton Symposium on Mathematical Programming in 1967 and appeared in its proceedings [14]. The new model, however, created hard convexity and computational problems. I knew from reliability theory and other earlier literature, that the logconcavity of univariate probability densities carries over to distribution functions, so I tried to prove its multivariate counterpart. What I ultimately proved was more than that. I introduced the concept of logarithmically concave (logconcave) measure (if A, B are convex subsets of R^n and $0 < \lambda < 1$, then $P(\lambda A + (1-\lambda)B) \geq P^{\lambda}(A)P^{1-\lambda}(B)$ and proved that if a probability measure P is generated by a logconcave density then P is logconcave [15]). I also proved that if a function of two sets of variables is logconcave, and is integrated with respect to one of the variable set, then the resulting function is logconcave in the other variable set [17]. These results were proved in the stochastic programming framework but are also celebrated in statistics, physics, convex geometry, economics, military science, sociology, etc. My first more detailed publication on probabilistic constraints are [14, 16, 30]. In [16] I introduced constraints of conditional expectation type, as well.

Abe Charnes was impressed by my works and nominated me foreign corresponding member of the National Academy of Engineering of Mexico. When he informed me about his action he wrote in his letter: "To set the nomination into perspective I am also nominating Leonid Kantorovich a foreign member". I received the honor in 1977. Kantorovich was not granted exit visa, could not come to the ceremony in Mexico. Two years later I became a corresponding member and in 1985 a full member of the Hungarian Academy of Sciences.

Subsequently, I formulated and solved, together with my students, several real life practical problems, where programming under probabilistic constraint was applied: five-year plan of the Hungarian power industry; flood control reservoir system design; inventory control; optimal water level regulation of Lake Balaton, etc [26, 27, 29, 30]. We have used multivariate normal, gamma and Dirichlet distributions.

Programming under probabilistic constraint with multivariate continuous distributions, however, has not become as popular as it deserves. In 1990 I started to use multivariate discrete distributions, introduced the concept of p-Level Efficient Point (pLEP), or p-efficient point [22], and the probabilistic constrained literature suddenly multiplied. My further papers in this respect were: Prékopa, Vizvári, Badics [33], where algorithmic generation of the p-efficient points and solutions of the probabilistic constrained problems with discrete random variables are given; Dentcheva, Prékopa, Ruszczyński (DPR) [8], where we present another solution to the problem in such a way that the p-efficient points are simultaneously generated with the algorithm. In the paper [25] I combine the supporting hyperplane method of Veinott and Szántai [34] with the DPR algorithm and solve efficiently the continuous variable problem. To find a new p-efficient point DPR propose the use of the knapsack problem. In a recent paper Prékopa, Unuvar [32] solve a probabilistic constrained network design problem, where the DPR algorithm is applied but the knapsack problem is solved in a special way for a random demand vector with independent components with discrete, logconcave marginal distributions. The above-mentioned algorithms work efficiently.

Probabilistic constraints can play an important role in dynamic problems too, one early network planning model of mine [18] shows the way. The difficulty is not that much in the numerical solution (even though it needs special effort) but in the statistical side of the application: we need satisfactory approximations to the conditional joint probability distributions of the future random variables, given past, present and future values of others.

4. Approximation of probabilities by linear programming

One of the important problems in programming under probabilistic constraint is the calculation of the values of multivariate c.d.f.'s and their gradients. Fortunately, the gradients of quite a few c.d.f.'s can be obtained by the same method (subroutine) that calculates the distribution function values. So, I had in mind to use methods that provide us with sharp lower and upper bounds, close to each other, that serve as approximations to probabilities of Boolean function of events.

The probability to approximate was intersection of events, which can be calculated by the use of the probability of the union of the complementary events. I knew the results by Dawson and Sankoff [7] and others for bounding the probability of the union, using some of the $S_i,\ i = 1, \ldots, n$ from the inclusion-exclusion formula. The latter cannot be used if n is large because only a few low subscripted S_i can be calculated. The existing formulas did not provide me with close bounds, so I turned to a classical formula that established a linear relationship between the $S_i,\ i = 1, \ldots, n$ and the probabilities of occurrences of the n events. Taking the first few equations with $S_1, \ldots, S_m (m \ll n)$ on the right hand side and using the probability that at least one event occurs, as objective function, to be both minimized and maximized, I succeeded to obtain a method based on linear programming to do the job. I proved that the problem is totally positive meaning that the determinants of all $m \times m$ submatrices of the matrix of equality constraints are positive and so are all $(m+1) \times (m+1)$ submatrices of the former one, supplemented by the coefficients of the objective function on top of it. This allowed me to present an elegant characterization of the dual feasible bases and to use it in a specialization of Lemke's dual algorithm, to solve the problem. In my algorithm, however, numerical work is required only to find the outgoing vector but the pricing in step is simple, we only have to restore the dual feasible structure of the basis for which there is always only one possibility. In addition to the algorithmic solution to the LP, closed form formulas are easy to obtain, using only S_1, S_2, S_3, S_4. With these I discovered that the sharp (sometimes called Bonferroni-type) inequalities are optimum values of discrete moment problems [21]. Earlier inequalities of this type are either special cases of the new ones or are not sharp.

The method of linear programming has already been used by Kwerel [11] but only to obtain bounds for the union using S_1, S_2, S_3.

Later I extended my research to obtain bounds for expectations and probabilities of other Boolean functions of events [21], introduced multivariate discrete moment problems and obtained bounds for probabilities and expectations. Another way to look at probability bounding is the use of the Boolean scheme [9], where the joint probabilities of the events are used individually, not only in sums of them as it is the case in $S_i,\ i = 1, \ldots, n$. The Boolean scheme is a disaggregated problem, as compared to the binomial moment problem, so the bounds with it are better. However, only partial results are known regarding the characterization of the dual feasible bases while full description exists in the other one. Some of my results concern the Boolean scheme (see, e.g., [3]).

It is interesting to remark that probability bounds can be used to solve deterministic problems. Boros and Prékopa [2] presented efficient solution for the satisfiability problem by randomizing the Boolean variables and using binomial moment upper bounds to obtain the result.

Discrete moment problems have become important tools in telecommunication, transportation and other network reliability calculation, and are also used in PERT, insurance, finance, numerical analysis, and other problems. Some of my current projects include joint works with my students: Jinwook Lee, Mariya Naumova, Anh Ninh and Kunikazu Yoda.

5. Applications

I carried out many applications of various types. They belong to one of the following groups: Inventory control, production, manufacturing; water resources; power systems; telecommunication and transportation; economics and finance; natural sciences, medicine and nutrition. Below I briefly describe some of my applied works, not mentioned in former parts of the paper.

Solution of a problem in connection with fiber glass manufacturing. The problem was to set up the manufacturing goals for the production, where the yields and the demands are random. Published with my former student M. Murr [12].

Reliability calculation of interconnected cooperating power systems. A general methodology, based on network analysis and bounding by the use of the binomial moment problem, was worked out by A. Prékopa and E. Boros. It was used with real life data for the calculation of the reliability of a Northern Ohio-Pennsylvania interconnected power system. The work was done by S.L. Fanelli under my leadership (Master Thesis, S. L. Fanelly, RUTCOR, 2012).

Optimal short term scheduling of the power generation with thermal power plants. The work was done in Hungary between 1975-1985 for the Hungarian power system but the model is formulated in general terms and applicable elsewhere, too. The main novelties of the model are: 1. It is not simplified, the physics of the AC system is fully used. 2. The unit commitment and the power transportation problems are put together in a large scale optimization problem, with complex, continuous and partly $0-1$ variables. 3. The problem is of Benders decomposition type and was solved by Benders decomposition and heuristics. Nonconvex, nonconcave, nonlinear constraints are also present in the problem. The developed code solved the nationwide problem in two minutes, allowing for efficient dispatch in case of the occurrence of an unforseen event. The 150 page report is available in English and soon will be submitted for publication in book form [28].

Five-year plan of the Hungarian power industry. The extensive industrial development in the country around 1970, called for a planning model in the power industry: how much to invest into the different types of power plants, how much change in the use of different fuel types is needed, taking into account the import, export possibilities and requirements in the course of a period of five years, to be able to supply the demands at minimum cost and large reliability. The import-export variables were assumed to be random with joint normal distribution. It has four components and its parameters were estimated from past history. A probabilistic constrained model, termed STABIL, was formulated with more than one hundred deterministic constraints and about fifty variables. At the beginning of the 1970s it was a computationally complex problem especially because of the presence of a probabilistic constraint involving multivariate normal distribution. We were the first to numerically solve problems of this kind. The results were very interesting. When we solved the problem, where we replaced the random variables with their expectations and then plugged in the optimal solution into the constraining function, the joint probability of the four stochastic constraints, turned out to be 0.1. On the other hand, the optimal objective function value was not much larger when we imposed a probabilistic constraint with lower bound $p = 0.9$. Thus, for a relatively small additional cost high reliability could be ensured. However, the optimal solution was different, significant changes could be observed in the various types of power plants [26].

Exact analytic solution for the dynamic programming model of the value of the Bermudan and American options. I derived complicated analytic formulas and my former student and colleague, T. Szántai supplied the calculation of the multivariate c.d.f. values along with the other numerical works. We carried out extensive calculation, presented a large number of numerical examples and concluded that the binomial tree and some similar methods, widely used in practice, overestimate the value of the American options. We received immediate reaction from specialists of the field but then explained that the binomial (or even the multinomial) tree method essentially uses the upper bound, at each step, in Jensen's inequality [31].

A model in behavioral science. Together with my former Ph.D. student, Mária Kopp (later professor and director of the Institute of Behavioral Science of the Semmelweis University of Medicine in Budapest), I created a game theoretical model for human behavior. Kopp wanted to learn statistics and game theory, that was her motivation to become my student. We came up with a novel principle: human beings want to minimize the loss of competence under changing circumstances surrounding them. The principle was also supported by Bayesian type statistical decision theoretical model but perhaps not enough convincing because the English version of the paper was not accepted for publication. Ten years later, however, the famous American psychologist Albert Bandura presented about the same principle under the name: self-efficacy, in an international journal. Our paper appeared in a Hungarian medical journal [10].

6. History of optimization

One of my discoveries in the history of mathematics concerns the history of optimization theory. Kuhn, Tucker (1951) proved a fundamental theorem in nonlinear programming in which there is a necessary condition for optimality of a multivariate function subject to inequality constraints. In addition to the assumptions on differentiability of the objective and constraining functions, they needed a regularity condition, called constraint qualification, for the proof. Later on it turned out that in a master thesis Karush, in 1939 at the University of Chicago, obtained essentially the same result. Today the theorem is called Karush, Kuhn, Tucker (KKT) theorem.

Farkas was Hungarian, professor of theoretical physics at the University of Kolozsvár (now Cluj-Napoca) in Transylvania (part of Hungary at that time) and a member of the H.A.S. I discovered that Farkas needed his inequality theorem for the same purpose as Kuhn and Tucker did. Farkas was working on the problem of mechanical equilibrium but unlike Lagrange (who had equality constraints) he had inequality constraints on the system. In case of a system of conservative forces potential exists and if it takes its minimum, then the system is in equilibrium. The nonlinear programming problem was a special case of Farkas' problem and here is why Farkas created the theory of linear inequalities: to prove the necessary conditions for equilibrium. Farkas' work was done in the framework of analytical mechanics, and, failed to include constraint qualification, his proof is incomplete from the mathematical point of view (so is, however, Lagrange's proof for the case of equality constraints). My paper on the development of optimization theory appeared in English [19].

7. Acknowledgement

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My Path through Combinatorial Optimization

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It is a great honor to be the recipient of the 2012 Farkas prize for mid-career researchers in optimization. I am very thankful to whoever nominated me for this prize, and I am especially grateful to the committee who selected me. The committee was chaired by David Shmoys and included Monique Laurent, Laurence Wolsey and Yinyu Ye. Thank you, Monique, David, Laurence and Yinyu. I should emphasize that the great body of work by the members of this committee has shaped my view of optimization and has had a big influence on my own research. With another composition of the prize committee, I would never have received this prize! In this article, I decided to make a few (boring, I am afraid) remarks regarding how I started working on the traveling salesman problem and semidefinite programming. For those who nevertheless decide to continue reading, take a glass of wine (or a Belgian beer) and relax.

First, how did I get interested in optimization, and especially combinatorial optimization? Well, since an early age, I have always enjoyed mathematics, its rigor and its beauty. It was rather obvious this was my main (and only?) strength. After finishing high school, I started an engineering degree at UCL (in Louvain-La-Neuve, Belgium) as this was the typical path for mathematically inclined youth in Belgium; for example, my Belgian teammates at the International Math Olympiads all went to Engineering school. As an undergrad, I majored in applied mathematics and I got hooked on combinatorial optimization thanks to one person, Laurence Wolsey, with whom I took two classes. I hope he won't read this article, as he is so modest and does not like to be placed in the spotlight, but let me nevertheless take this opportunity to celebrate (take a sip!) his very well-deserved 2012 Dantzig prize (awarded for "original research, which by its originality, breadth, and scope is having a major impact on the field of mathematical optimization") and 2012 John von Neumann Theory Prize (awarded for "fundamental, sustained contributions to theory in operations research and management sciences"). Laurence also was the one who suggested I do my Ph.D. in the US.

What I find particularly appealing to combinatorial optimization is that it is a discipline with numerous, important, industrial applications, but also with two faces: A beautiful, reasonably well-understood mathematical foundation of 'easy', tractable problems, and a vast array of open mathematical questions and computational challenges for the hard, intractable problems.

I can still visualize the lecture in which Laurence described Christofides' algorithm for the symmetric traveling salesman problem (STSP). He mentioned that a major open problem was to find an algorithm for STSP that improves upon Christofides' algorithm in the worst-case. I stopped listening, and for the rest of the lecture, tried a few, naive ideas. And, now, more than 25 years later, I am still trying to obtain such an improved approximation algorithm for the STSP but, hopefully, with a (slightly) deeper understanding of the problem... My interest in the approximability of the TSP also got reinforced by David Shmoys, who was on the faculty at MIT during my graduate studies there. Unfortunately, I was too shy and not confident enough in my abilities to work under his supervision. But David also mentioned to me the conjectured worst-case behavior of the Held-Karp lower bound for STSP obtained by optimizing over the subtour polytope. In a 1980 paper, Wolsey [17] had proved that the cost of the optimum tour (or even Christofides' tour) is never greater than 3/2 times the Held-Karp lower bound, but it is widely believed that the worst-case gap should give a factor of 4/3 (although the 3/2 bound is tight for Christofides' algorithm). Over the last 25 years, this open question has somehow motivated several results of mine, including (i) work on the probabilistic analysis [9] of the Held-Karp lower bound (when points are independently and uniformly generated in the unit square), (ii) a framework for assessing the strength of cutting planes for combinatorial optimization problems [12] and illustrated on the STSP, (iii) an approximate algorithm for the minimum spanning tree problem under degree restrictions [8], (iv) a construction for instances of the Asymmetric TSP (ATSP) achieving an integrality gap arbitrarily close to 2 for the asymmetric version of the Held-Karp lower bound [4], and (v) a better than logarithmic approximation algorithm for ATSP [3].

Let me briefly discuss two of these TSP-related results. In [12], I have shown that the addition to the subtour polytope of essentially all well-known classes of valid inequalities for the TSP (including clique tree and path configuration inequalities) will not increase its optimum value by more than a factor of 4/3. This provides some support to the conjectured integrality gap of 4/3 for the subtour relaxation. The proof idea is elementary, but does not apply immediately to implicitly defined families of valid inequalities, such as the widely successful local cuts of Applegate et al. [2]. Among these TSP-related results, my favorite one is about minimum cost spanning trees under degree restrictions [8]. Held and Karp, in their landmark paper [13] on the traveling salesman problem, have shown that any solution to the subtour polytope can be viewed as a convex combination of 1-trees, i.e. a spanning tree on $V \setminus \{1\}$ together with two edges incident to vertex 1. In 1991, while studying the worst-case integrality gap of the subtour relaxation, I conjectured that one could restrict to 1-trees having maximum degree 3 in the convex combination. This naturally lead to investigating minimum cost spanning trees under the additional constraints that every vertex has degree at most k. Note that even deciding if a spanning tree of maximum degree at most k exists in a given graph is NP-hard. At the time, I was able to solve an approximate version without costs, and derive an algorithm that either constructs a spanning tree of maximum degree at most $k+1$ or obtain a certificate that no spanning tree of maximum degree k exists. The algorithm was reminiscent of Edmonds' algorithm for matroid intersection, but I did not publish it then as I was convinced that a similar result was possible for the problem with costs. As it happens, Fürer and Raghavachari [7] independently found a different algorithm for the version without costs. But, 15 years later, I was quite happy to be able to tackle the cost version and give an algorithm that produces a spanning tree of maximum degree at most $k+2$ whose cost is at most the cost of the minimum cost spanning tree of degree at most k [8]; this is equivalent to showing that every solution to the spanning tree polytope with fractional degrees at most k can be viewed as a convex combination of spanning trees of maximum degree at most $k+2$. This is done by proving properties of extreme points of the corresponding LP relaxation by algebraic means, by defining a related matroid intersection problem whose common bases are (some, not all) spanning trees of degree at most $k+2$, and using the knowledge of the matroid intersection polytope. This result was improved by Singh and Lau [16] by completely different means using iterative rounding; they were able to relax the maximum degree bound to $k+1$ rather than $k+2$. Going back to 1-trees, this shows that any solution to the subtour polytope can be viewed as a convex combination of 1-trees of maximum degree 3, but it

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is still unclear whether this is of any use for charaterizing the worst-case integrality gap.

My receiving the Farkas prize is probably mostly due to my most celebrated and cited result, an approximation algorithm for the maximum cut problem (MAXCUT) [11] based on semidefinite programming, and this was obtained jointly with my first Ph.D. student, David Williamson. This is the right time to mention that, over the years, I have been extremely fortunate to interact with a great number of talented Ph.D. students including David, and this has been hugely stimulating for me in my ways, and I sincerely hope for them too. I would like to explain how the MAXCUT result came to exist. I could start by quoting Don Knuth in a delightful article about mathematical license plates [14]:

> Sometimes people obtain mathematically significant license plates purely by accident, without making a personal selection. A striking example of this phenomenon is the case of Michel Goemans, who received the following innocuous-looking plate from the Massachusetts Registry of Motor Vehicles when he and his wife purchased a Subaru at the beginning of September 1993:

> Two weeks later, Michel got together with his former student David Williamson, and they suddenly realized how to solve a problem that they had been working on for some years: to get good approximations for maximum cut and satisfiability problems by exploiting semidefinite programming. Lo and behold, their new method — which led to a famous, award-winning paper [11] — yielded the approximation factor .878! There it was, right on the license, with C, S, and W, standing respectively for cut, satisfiability, and Williamson.

Although completely true (with some interpretation for CSW, of course), this does not tell the story that led to the discovery. More than four years earlier, in the summer of 1989, I did an internship in the group of David Johnson at Bell Labs in Murray Hill, NJ, and I participated in a DIMACS workshop on Polyhedral Combinatorics organized by Bill Cook and Paul Seymour. This was an incredibly valuable experience. Laci Lovász and Lex Schrijver each gave a series of lectures, and this is where I heard about their so-called matrix cuts [15]. For a graduate student who had read several of their books and articles, this was a real treat. They defined two operators N and N_+ which automatically provide a strengthening of any convex set towards its $0-1$ integer hull, and by applying them repeatedly one obtains a hierarchy of relaxations converging to the integer hull in at most n iterations, where n is the dimension. And the stronger of the two operators, N_+, involves imposing semidefinite constraints. That was my first exposure to *semidefinite programming*. This was fascinating, and I spent the rest of the summer trying to solve the many open questions they raised, with little success. But I was completely convinced of the power of semidefinite programming, especially after devouring Lovász's 1979 paper on the Shannon capacity and results on perfect graphs, and I wanted to find other applications of it. I learned much about semidefinite programming from an early draft (first submited in 1991) of a paper by Farid Alizadeh [1] in which he laid much of the foundation of semidefinite programming (in terms of conic programming and duality) and he extended Yinyu Ye's projective method for linear programming to semidefinite programming. My goal was to find a problem for which a natural semidefinite programming relaxation is provably stronger than any natural linear programming relaxation, and I quickly focused on the obvious candidate, the maximum cut (MAXCUT) problem (i.e., unconstrained quadratic $0-1$ programming with special costs), as the semidefinite programming relaxation seemed particularly easy to describe and handle. However, proving a worst-case bound for the relaxation appeared quite challenging and, over the years, I tried first alone and then with David Williamson many different approaches, often involving the dual semidefinite program (which can be nicely expressed as a vector subset sum problem). This was also the time that David and I were working on our primal-dual linear programming approaches [10] to design approximation algorithms for a class of network design problems. We realized that our semidefinite programming bound was equivalent to an eigenvalue bound which had just been considered by Delorme and Poljak [5], and which they conjectured to be within 0.88445 of the optimum; this value corresponds to the gap for the 5-cycle. But after putting the maximum cut problem aside many times and always being drawn again to it a few months later, we finally managed to find the simplest possible and most natural approach to get a cut provably within a factor of $\min_x \frac{2\arccos(x)}{\pi(1-x)} \sim 0.878$ of the value of the optimum cut. How we could we have missed this before? The ingredients of our approach are elementary and it appears that some had been considered before. The random hyperplane technique and its analysis are indeed tighly related to ℓ_1-embeddability of the *spherical distance space* as discussed in Section 6.4 of the encyclopedic book [6] on cuts and metrics by Michel Deza and Monique Laurent. What I find most amazing about our MAX-CUT result is that it appears in many ways to stand right on the ridge between the two (easy and hard) faces of combinatorial optimization.

After reminiscing about some of my research, I would like to quote one concluding sentence from the embarassingly laudatory citation for this Farkas prize:

> Goemans's contributions are marked by his innate ability to give the elegant solution; in fact, his knack of finding the beautiful proof ("from the book") has led to a much deeper understanding of the work of many others as well.

This is definitely exagerated, although I am indeed typically obsessed with finding the right, naturally simple and elegant proof, and I am never satisfied (as my students can unfortunately attest) with the first approach or solution that comes up. And I hope to continue exploring the beautiful scenery in and around combinatorial optimization for the second half of my career.

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Using the Pythagorean Theorem in a Polynomial Algorithm for Linear Programming

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The Pythagorean theorem is a fundamental property of Euclidean spaces. One of its numerous and well-known consequences is the following equation that expresses the relationship between the length h of the altitude of a right triangle in $\mathbb{R}^n$ (the altitude to its hypotenuse) and the lengths a and b of its legs:

$$\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

The purpose of this short article is to outline a polynomial algorithm for linear programming based on exactly this formula. The algorithm has some distinctive features that we discuss at the end of the article. The full version of the algorithm is given in the manuscript [3]. The new algorithm improves the previous algorithm presented in [2].

In a certain sense, our algorithm has connections with the relaxation method for systems of linear inequalities of Agmon [1] and Motzkin and Schoenberg [4]. The relaxation method constructs a sequence of points $x^{(s)}$ that follows a simple recursive rule. In each iteration, among the constraints of a system $Cx \geq d$ (C is a matrix and d is a column vector) the relaxation method determines a constraint $c^T x \geq \delta$ that is violated by the current point $x^{(s)}$. The next point has the form

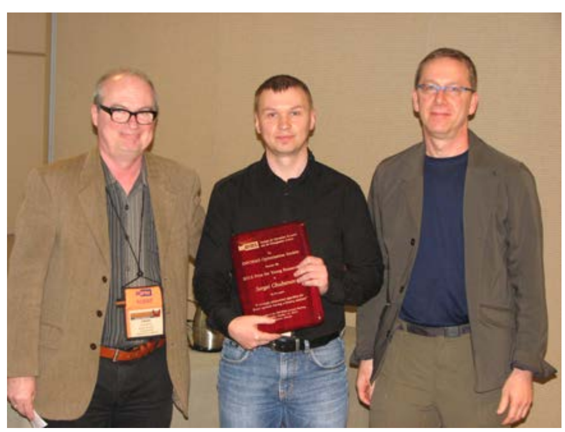
$$x^{(s+1)} = x^{(s)} + \lambda c.$$

An appropriate choice of the parameter λ ensures that the sequence of points converges to a solution of the system. In particular, λ can be chosen so that $x^{(s+1)}$ is the orthogonal projection of $x^{(s)}$ onto the half-space $\{x | c^T x \geq \delta\}$. In fact, the relaxation method belongs to the class of alternating projection methods such as for instance von Neumann's and Dijkstra's projection algorithms.

Every linear program can be reduced in polynomial time to the problem of finding a solution of a system $Ax = \mathbf{0}, x > \mathbf{0}$. (We will not explain the notation because it is conventional and intuitively clear. All the coefficients are assumed to be rational.) That is, we are looking for a solution with positive components. It should be noted that our method would remain almost unchanged if we only required that some components are positive and the other are nonnegative.

It is convenient to consider the associated system $Cx > \mathbf{0}$ where C is the matrix of the orthogonal projection onto the linear subspace generated by the equations $Ax = \mathbf{0}$ (that is, onto the subspace $\{x | Ax = \mathbf{0}\}$). If x^* is a solution of $Cx > \mathbf{0}$, then Cx^* is a solution of the original system because $ACx^* = \mathbf{0}$ due to the fact that Cx^* is the projection of x^* onto the subspace.

Now we apply a procedure that we call the basic procedure. As well as the relaxation method, it constructs a sequence of points following a simple recursive rule. Let $c^T x > 0$ be a constraint, of the system $Cx > \mathbf{0}$, violated by the current point $x^{(s)}$. (If no one of the constraints is violated, then $Cx^{(s)}$ is a feasible solution.) An iteration of the ba-



Endre Boros, Sergei Chubanov and Jon Lee sic procedure consists in taking a step towards the point c. More precisely, the basic procedure chooses a point $x^{(s+1)}$ on the segment $[x^{(s)}, c]$ with the minimum norm among the points of the segment. This means that, for some coefficient α in $[0, 1]$,

$$x^{(s+1)} = \alpha x^{(s)} + (1 - \alpha)c.$$

This obviously resembles the formula used by the relaxation method. However, what we obtain, and what cannot be obtained by the relaxation method, is the performance guarantee in terms of the reciprocals of the norms of the constructed points:

$$\frac{1}{\|x^{(s+1)}\|^2} \geq \frac{1}{\|x^{(s)}\|^2} + 1.$$

To obtain this inequality, we need to note that, because $c^T x > 0$ is violated by $x^{(s)}$, the triangle formed by the points $x^{(s)}$, $\mathbf{0}$, and c contains a right triangle with the right angle at $\mathbf{0}$. The altitude of the right triangle is the segment $[\mathbf{0}, x^{(s+1)}]$. Using the formula for the altitude of a right triangle and remembering that $\|c\| \leq 1$ because C is the projection matrix, we derive the above inequality.

We choose the initial point $x^{(0)}$ as a convex combination of the columns of C. Then the obtained points $x^{(s)}$ are convex combinations of the columns of C. These convex combinations are explicitly constructed in the course of the procedure and play a key role in its analysis. The above inequality guarantees that we need at most $O(n^3)$ iterations until the fulfillment of the condition

$$\|x^{(s)}\| \leq \frac{1}{2n\sqrt{n}}.$$

(The n denotes the number of variables.) Now we can use the fact that, as a convex combination of the columns of C, the current point $x^{(s)}$ has the form $x^{(s)} = Cy^T$ where y is a nonnegative row vector with $\sum_i y_i = 1$. As C is symmetric, $x^{(s)} = (yC)^T$. Let x^* be a solution of $Ax = \mathbf{0}$ in the unit cube $[0, 1]^n$. (It follows that $\|x^*\| \leq \sqrt{n}$.) Since $Ax^* = \mathbf{0}$, we have $Cx^* = x^*$. (That is, the projection of x^* onto the linear subspace is the same vector x^* because x^* lies in this subspace.) Using the Cauchy-Schwarz inequality, we write

$$yx^* = yCx^* = (x^{(s)})^T x^* \leq \|x^{(s)}\|\|x^*\|.$$

This immediately implies that

$$yx^* \leq \frac{1}{2n}.$$

Now we consider a component y_k with $y_k = \max\ y$ and note that $y_k \geq \frac{1}{n}$. The above upper bound on yx^* implies that

$$x^*_k \leq \frac{1}{2}.$$

This holds for every solution x^* of $Ax = \mathbf{0}$ in the unit cube. Now we transform A by dividing the kth column of A by 2, which can be viewed as scaling along the axis x_k by the scale factor of 2. If the system $Ax = \mathbf{0}, x > \mathbf{0}$, is feasible, then it has a solution x^* in $[0, 1]^n$. Let ϕ be a mapping that maps x^* to the point obtained from x^* by multiplying x^*_k by 2. The point $\phi(x^*)$ belongs to the unit cube because of the upper bound on x^*_k. Moreover, $\phi(x^*)$ is a feasible solution of the transformed system. That is, one can say that all feasible solutions in the unit cube remain in the unit cube under the transformation of A. We construct the projection matrix C with respect to the transformed matrix A and apply the basic procedure to the system $Cx > \mathbf{0}$. Note that if there is a solution of the original system then there is one whose binary sizes of components are bounded by a value L that is polynomially bounded in the size of the system. Such a bound L can easily be found. Let us repeat the calls to the basic procedure followed by the respective transformations of the matrix A. Note that if the original system is feasible then the basic procedure must find a feasible solution to some of the transformed systems in at most nL calls. (The obtained solution is then translated back to a feasible solution of the original system.) If this is not the case, we conclude that the original system is infeasible. Since the basic procedure runs in polynomial time, we have a polynomial algorithm (we need of course an additional technique to control the sizes of the numbers).

Whenever we prove that $x^*_k \leq \frac{1}{2}$ for every solution x^* of $Ax = \mathbf{0}$ in the unit cube, we can conclude that x^*_k is zero in all 0-1 solutions of this system. In place of dividing by 2, we now delete the respective column of A and call the basic procedure again. This leads to

Theorem 1 *Consider a system*

$$Ax = b, \mathbf{0} \leq x \leq \mathbf{1}, \qquad (1)$$

where A is a rational matrix and b is a rational column vector. There is a strongly polynomial algorithm, with the running time $O(n^4)$, that either finds a solution of this system or proves that the system has no integral solutions.

Proof. Observe that one can construct a system $A'x' = \mathbf{0}, x' \geq \mathbf{0}$, with $2n+1$ variables, having a nontrivial 0-1 solution if and only if the system (1) has a 0-1 solution. To either find a solution or prove that no nontrivial 0-1 solution exists, we need at most $O(n)$ calls to the basic procedure. The running time $O(n^4)$ is obtained by a further analysis and modification of the algorithm (see [3]), which allows to control the initial data for the basic procedure so that to prevent unnecessary steps. ∎

So one of the following two results can be achieved in strongly polynomial time: Either a feasible solution of (1) (with maybe fractional components) or a proof that no 0-1 solution exists. In the latter case we solve the respective instance of the NP-hard problem of finding a solution of $Ax = b, x \in \{0,1\}^n$.

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A Strong Dual for Conic Mixed-Integer Programs

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1. Introduction

Duality is crucial in understanding theoretical properties of optimization problems and often plays a key role in the development of algorithms to find the corresponding optimal solutions.

For a minimization problem $(\mathcal{P})$, a *(weak) dual* is a maximization problem $(\mathcal{D})$ such that the objective function value of any of its feasible solutions gives a lower bound on the objective function value of any feasible solution of $(\mathcal{P})$. Upper bounds on the objective function value of the primal $(\mathcal{P})$ are provided by primal feasible solutions. We say that a minimization problem is *finite* if its feasible region is nonempty and the objective function is bounded from below. A *strong dual* $(\mathcal{D})$ for $(\mathcal{P})$ is a (weak) dual satisfying the following additional two properties:

1. $(\mathcal{P})$ is finite if and only if $(\mathcal{D})$ is finite.

2. If $(\mathcal{P})$ is finite, then the optimal objective function values of $(\mathcal{P})$ and $(\mathcal{D})$ are the same.

In the case of linear programming and, more generally, conic programming, duality is well understood (see, for instance, [1]). The subadditive dual for mixed-integer linear programs has also been widely studied ([3, 4, 5, 6, 7, 8, 11]). However, such a result for their mixed-integer conic counterpart was not known in the literature. In our paper [10], we take a step further in the understanding of duality for problems with particular structure by extending the subadditive dual for mixed-integer linear programs to the case of mixed-integer conic programs.

2. A Little Bit of Notation

Let $K \subseteq \mathbb{R}^m$ be a full-dimensional, closed and pointed cone. For $a, b \in \mathbb{R}^m$, we denote $a \succeq_K b$ if and only if $a - b \in K$. In addition, we write $a \succ_K b$ whenever $a - b \in \text{int}(K)$.

A mixed-integer conic programming problem (the primal optimization problem) is an optimization problem of the following form:

$$(\mathcal{P}) \begin{cases} \inf & c^t x \\ \text{s.t.} & Ax \succeq_K b \\ & \quad x_i \in \mathbb{Z}, \forall i \in \mathcal{I}, \end{cases}$$

where $\mathcal{I} = \{1, \dots, n_1\} \subseteq \{1, \dots, n\}$ is the set of indices of integer variables.

Notice that mixed-integer linear programming problems are a special case of problems of the form of $(\mathcal{P})$, by setting $K = \mathbb{R}^m_+$. Hence, a natural way of defining a dual optimization problem for mixed-integer conic programming is to generalize the well-known subadditive dual for mixed-integer linear programming (see, for example, [3] and [12]).

Let $\mathcal{S} \subseteq \mathbb{R}^m$. A function $g : \mathcal{S} \mapsto \mathbb{R} \cup \{-\infty\}$ is said to be *subadditive* if for all $u, v \in \mathcal{S}$ such that $u + v \in \mathcal{S}$, the inequality $g(u+v) \leq g(u) + g(v)$ holds. The function g is said to be *nondecreasing w.r.t.* K if for all $u, v \in \mathcal{S}$ such that $u \succeq_K v$, the inequality $g(u) \geq g(v)$ is satisfied.

We define the subadditive dual problem for $(\mathcal{P})$ as

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follows:

$$(\mathcal{D}) \begin{cases} \sup \ g(b) \\ \text{s.t. } \mathrm{g}\left(\mathrm{A}^{\mathrm{i}}\right) = -\mathrm{g}\left(-\mathrm{A}^{\mathrm{i}}\right) = \mathrm{c}_{\mathrm{i}}, \quad \forall \mathrm{i} \in \mathcal{I} \\ \qquad \bar{g}\left(A^i\right) = -\bar{g}\left(-A^i\right) = c_i, \quad \forall i \notin \mathcal{I} \\ \qquad g(0) = 0 \\ g : \mathbb{R}^m \to \mathbb{R} \text{ is subadditive and} \\ \text{nondecreasing w.r.t. to } K, \end{cases}$$

where A^i denotes the ith column of A and for a vector $d \in \mathbb{R}^m$ we denote $\bar{g}(d) := \limsup_{\delta \to 0^+} \frac{g(\delta d)}{\delta}$.

3. Main Result

In the theory of subadditive duality for mixed-integer linear programming ($K = \mathbb{R}^m_+$), a sufficient condition to have strong duality is the rationality of the data defining the problem, that is, $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$. The main result of this paper is to show that strong duality for mixed-integer conic programming holds under the following technical condition:

There exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ *such that* $A\hat{x} \succ_K b \quad (*)$,

where $n_2 = n - n_1$.

Theorem 1 (Strong duality) *If there exists* $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ *such that* $A\hat{x} \succ_K b$, *then*

1. $(\mathcal{P})$ *is finite if and only if* $(\mathcal{D})$ *is finite.*
2. *If* $(\mathcal{P})$ *is finite, then there exists a function* g^* *feasible for* $(\mathcal{D})$ *such that* $g^*(b) = \inf\{c^T x \mid Ax \succeq_K b, \ x \in (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\}$.

4. Sketch of the Proof

Our proof of strong duality for mixed-integer conic programs is an adaptation of the ideas used to prove strong duality for mixed-integer linear programs (see [3, 12]).

4.1 Basic Ingredients

We give a short description of the crucial results that we use in the proof of Theorem 1.

1. **Properties of $\succeq_K$.** The vector inequality $\succeq_K$ defines a partial order relationship that satisfies: (1) The *homogeneity property*: for all $\lambda \geq 0$ if $u \succeq_K v$, then $\lambda u \succeq_K \lambda v$ and (2) The *additivity property*: if $u \succeq_K v$ and $u' \succeq_K v'$, then $u+u' \succeq_K v+v'$ (see [1]). These properties of $\succeq_K$ give a very nice structure to the feasible region of the primal problem.

2. **Strong Duality for Conic Programming.** The continuous relaxation of $(\mathcal{P})$, denoted by $(\mathcal{P}_R)$, is the problem obtained when in a problem of the form $(\mathcal{P})$ we set $\mathcal{I} = \emptyset$. It is well-known that if there exists $\hat{x} \in \mathbb{R}^n$ such that $A\hat{x} \succ_K b$, then $(\mathcal{P}_R)$ has a strong dual, denoted by $(\mathcal{D}_R)$, whose feasible points are linear functions that satisfy all the constraints in $(\mathcal{D})$.

3. **Weak Duality for $(\mathcal{P})$ and $(\mathcal{D})$.** It is easy to see from the definitions that $(\mathcal{D})$ is a weak dual for $(\mathcal{P})$, that is, for all $x \in \mathbb{R}^n$ feasible for $(\mathcal{P})$ and for all $g : \mathbb{R}^m \mapsto \mathbb{R}$ feasible for $(\mathcal{D})$, we have $g(b) \leq c^T x$.

4. **Finiteness Property for Convex Mixed-integer Programs.** Let $B \subseteq \mathbb{R}^n$ be a closed convex set such that $B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$. Consider the convex mixed-integer optimization problem $(\mathcal{Q}) := \inf\{c^T x \,|\, x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\}$ and its continuous relaxation $(\mathcal{Q}_R) := \inf\{c^T x \,|\, x \in B\}$. We say that $(\mathcal{Q})$ and $(\mathcal{Q}_R)$ satisfy the *finiteness property* if $(\mathcal{Q})$ is finite if and only if $(\mathcal{Q}_R)$ is finite.

   Notice that in the case of mixed-integer linear programming (the set B is a polyhedron defined by rational data) it is well-known that $(\mathcal{Q})$ and $(\mathcal{Q}_R)$ satisfy the finiteness property. However, in the case of mixed-integer conic programming (the set B is the feasible region of a problem of the form $(\mathcal{P}_R)$ and B is defined by rational data), then there are some examples that show that the finiteness property is not necessarily satisfied. In the following lemma, we present a sufficient condition for the finiteness property to hold in the case of general mixed-integer convex optimization problems.

   Lemma 1 ([10]) *If* $\mathrm{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$, *then* $(\mathcal{Q})$ *and* $(\mathcal{Q}_R)$ *satisfy the finiteness property (where* $\mathrm{int}(B)$ *denotes the interior of the set* B*).*

   We use Lemma 1 to show that if there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$, then $(\mathcal{P})$ is finite if and only if $(\mathcal{P}_R)$ is finite.

5. **Properties of the value function of $(\mathcal{P})$.** The value function of $(\mathcal{P})$, $f : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$, is defined as

$$f(u) = \inf\{c^T x \ : \ Ax \succeq_K u, \ x_i \in \mathbb{Z}, \ \forall\, i \in \mathcal{I}\}.$$

   The domain of f, denoted by Ω, is defined as all vectors $u \in \mathbb{R}^m$ such that the problem $(\mathcal{P})$ with r.h.s $b := u$ is feasible.

   We use an important property of the value function: it can be proven that, in general, f satisfies all the constraints of the dual $(\mathcal{D})$, except that Ω is not necessarily equal to $\mathbb{R}^m$.

4.2 Some cooking steps

We now give a brief outline of the proof.

Proving (1.) in Theorem 1:
We show each direction of the equivalence.

($\Rightarrow$) If $(\mathcal{P})$ is feasible and bounded, then by the Finiteness Property we obtain that $(\mathcal{P}_R)$ is also feasible and bounded. Thus, by Strong duality for Conic Programs, we obtain that $(\mathcal{D}_R)$ is feasible. Since the feasible solutions of $(\mathcal{D}_R)$ satisfy all constraints of $(\mathcal{D})$, we obtain that $(\mathcal{D})$ is feasible. Finally, since $(\mathcal{P})$ is feasible, by Weak Duality we conclude that $(\mathcal{D})$ is also bounded.

($\Leftarrow$) We prove that if $(\mathcal{P})$ is infeasible, then $(\mathcal{D})$ cannot be finite. The proof idea of this statement is as follows: The fact that $(\mathcal{P})$ is infeasible can be used to construct an 'recession direction' for the feasible region of $(\mathcal{D})$ with positive objective function value (see, for instance, [3]). Therefore, in the case $(\mathcal{D})$ is feasible, we conclude that it is unbounded.

Proving (2.) in Theorem 1:
We want to show that if $(\mathcal{P})$ is feasible and bounded, then there exists a function g^* feasible for $(\mathcal{D})$ such that $g^*(b) = f(b)$. If $\Omega = \mathbb{R}^m$, then, since in this case f is feasible for $(\mathcal{D})$, we can take

$g^* = f$. Otherwise, if $\Omega \neq \mathbb{R}^m$, we extend f to a function g^* such that g^* is feasible for the dual $(\mathcal{D})$ and $g^*(b) = f(b)$. This extension is very technical and explicitly uses the fact that there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$.

5. Applications

A valid inequality (a.k.a. cutting plane) for $(\mathcal{P})$ is a linear inequality that is satisfied by all feasible solutions of $(\mathcal{P})$. As a consequence of strong duality, we obtain that all valid inequalities can be written as or dominated by a linear inequality of the form

$$\sum_{i \in \mathcal{I}} g(A^i)x_i + \sum_{i \notin \mathcal{I}} \overline{g}(A^i)x_i \geq g(b),$$

where g is a feasible dual function.

From the practical point of view, cuttings planes are one of the central tools used by modern mixed-integer programming solvers (see, for example, [2, 9]). In the case of general mixed-integer linear programs (MILP's), one of the most successful class of cutting planes in practice are the Gomory mixed-integer cuts (GMI). It turns out that (GMI) cuts are given by a well-known family of dual feasible functions corresponding to 1-row (MILP's). Therefore, we can use certain families of dual feasible functions to generate very powerful cutting planes for solving (MILP's). We expect dual feasible solutions of mixed-integer conic programs to be used in similar applications.

6. Final Remarks

Notice that assumption $(*)$ plays a similar role as the assumption of rational data in the case of mixed-integer linear programs in the proof of the strong duality result. More precisely, both assumptions are a sufficient condition for the finiteness property to hold, and they are also crucial in the construction of the extension of the value function of $(\mathcal{P})$ (in the case $\Omega \neq \mathbb{R}^m$).

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Nominations for Society Prizes Sought

The Society awards four prizes, now annually, at the INFORMS annual meeting. We seek nominations and applications for each of them, due by **June 30, 2013**. Details for each of the prizes, including eligibility rules and past winners, can be found by following the links from `http://www.informs.org/Community/Optimization-Society/Prizes`.

Each of the four awards includes a cash amount of US$ 1,000 and a citation certificate. The award winners will be invited to give a presentation in a special session sponsored by the Optimization Society during the INFORMS annual meeting in Minneapolis, MN in October 2013 (the winners will be responsible for their own travel expenses to the meeting).

The **Khachiyan Prize** is awarded for outstanding life-time contributions to the field of optimization by an individual or team. The topic of the contribution must belong to the field of optimization in its broadest sense. Recipients of the INFORMS John von Neumann Theory Prize or the MPS/SIAM Dantzig Prize in prior years are not eligible for the Khachiyan Prize. The prize committee for the Khachiyan Prize is as follows:

- Jorge Nocedal (Chair)
  nocedal@eecs.northwestern.edu
- Michael Todd
- Jean-Philippe Vial
- Laurence Wolsey

Nominations and applications for the Khachiyan Prize should be made via email to the prize-committee chair. Please direct any inquiries to the prize-committee chair.

The **Farkas Prize** is awarded for outstanding contributions by a mid-career researcher to the field of optimization, over the course of their career. Such contributions could include papers (published or submitted and accepted), books, monographs, and software. The awardee will be within 25 years of their terminal degree as of January 1 of the year of the award. The prize may be awarded at most once in their lifetime to any person. The prize committee for the Farkas Prize is as follows:

- Dimitris Bertsimas (Chair)
  dbertsim@mit.edu
- George Nemhauser
- Yurii Nesterov
- Yinyu Ye

Nominations and applications for the Farkas Prize should be made via email to the prize-committee chair. Please direct any inquiries to the prize-committee chair.

The **Prize for Young Researchers** is awarded to one or more young researcher(s) for an outstanding paper in optimization that is submitted to and accepted, or published in a refereed professional journal. The paper must be published in, or submitted to and accepted by, a refereed professional journal within the four calendar years preceding the year of the award. All authors must have been awarded their terminal degree within eight calendar years preceding the year of award. The prize committee for the Prize for Young Researchers is as follows:

- Alper Atamtürk (Chair)
  atamturk@berkeley.edu
- Samuel Burer
- Andrzej Ruszczyński
- Nikolaos Sahinidis

Nominations and applications for the Prize for Young Researchers should be made via email to the prize-committee chair. Please direct any inquiries to the prize-committee chair.

The **Student Paper Prize** is awarded to one or more student(s) for an outstanding paper in optimization that is submitted to and received or published in a refereed professional journal within three calendar years preceding the year of the award. Every nominee/applicant must be a student on the first of January of the year of the award. Any coauthor(s) not nominated for the award should send a letter indicating that the majority of the nominated work was performed by the nominee(s). The prize committee for the Student Paper Prize is as follows:

- Simge Küçükyavuz (Chair)
  kucukyavuz.2@osu.edu
- Santanu S. Dey
- Guanghui Lan

Nominations and applications for the Student Paper Prize should be made via email to the prize-committee chair. Please direct any inquiries to the prize-committee chair.

Nominations of Candidates for Society Officers Sought

Jon Lee will complete his term as Most-Recent Past-Chair of the Society at the conclusion of the 2013 INFORMS annual meeting. Sanjay Mehrotra is continuing as Chair through 2014. Jim Luedtke will also complete his term as Secretary/Treasurer at the conclusion of the INFORMS meeting. The Society is indebted to Jon and Jim for their work.

We would also like to thank four Society Vice-Chairs who will be completing their two-year terms at the conclusion of the INFORMS meeting: Brian Borchers, Santanu Dey, Mohammad Oskoorouchi, and Baski Balasundaram.

We are currently seeking nominations of candidates for the following positions:

- Chair-Elect
- Secretary/Treasurer
- Vice-Chair for Computational Optimization
- Vice-Chair for Integer Programming
- Vice-Chair for Linear Programming and Complementarity
- Vice-Chair for Networks

Self nominations for all of these positions are encouraged.

To ensure a smooth transition of the chairmanship of the Society, the Chair-Elect serves a one-year term before assuming a two-year position as Chair; thus this is a three-year commitment. As stated in the Society Bylaws, "The Chair shall be the chief administrative officer of the OS and shall be responsible for the development and execution of the Society's program. He/she shall (a) call and organize meetings of the OS, (b) appoint ad hoc committees as required, (c) appoint chairs and members of standing committees, (d) manage the affairs of the OS between meetings, and (e) preside at OS Council meetings and Society membership meetings."

The Secretary/Treasurer serves a two-year term. According to Society Bylaws, "The Secretary/Treasurer shall conduct the correspondence of the OS, keep the minutes and records of the Society, maintain contact with INFORMS, receive reports of activities from those Society Committees that may be established, conduct the election of officers and Members of Council for the OS, make arrangements for the regular meetings of the Council and the membership meetings of the OS. As treasurer, he/she shall also be responsible for disbursement of the Society funds as directed by the OS Council, prepare and distribute reports of the financial condition of the OS, help prepare the annual budget of the Society for submission to INFORMS. It will be the responsibility of the outgoing Secretary/Treasurer to make arrangements for the orderly transfer of all the Society's records to the person succeeding him/her."

Vice-Chairs also serve a two-year term. According to Society Bylaws, "The main responsibility of the Vice Chairs will be to help INFORMS Local Organizing committees identify cluster chairs and/or session chairs for the annual meetings. In general, the Vice Chairs shall serve as the point of contact with their sub-disciplines."

Please send your nominations or self-nominations to Jim Luedtke (jrluedt1@wisc.edu), including contact information for the nominee, by Saturday, June 1, 2013. Online elections will begin in mid- August, with new officers taking up their duties at the conclusion of the 2013 INFORMS annual meeting.

Host for the 2014 INFORMS Optimization Society Conference Sought

The fifth INFORMS Optimization Society Conference will be held in early 2014. The most recent OS conference, held in 2012 at the University of Miami, was a great success, offering an opportunity for researchers studying optimization-related topics to share their work in a focused venue. The Optimization Society is currently seeking candidate locations to host the 2014 conference. If you are interested in helping to host the conference, please contact the Optimization Society chair, Sanjay Mehrotra (mehrotra@northwestern.edu), by April 15, 2013.