Multistage Air Traffic Flow Management Under Capacity Uncertainty: A Robust and Adaptive Optimization Approach

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Outline

Motivation: Impact of Weather

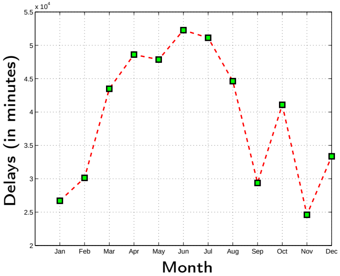

Figure: OPSNET[1] monthly delays for 2009.

- Increase in delays during the **summer months**.
- Thunderstorms account for **majority of air traffic delays** (60-70% monthly).

[1] The Operations Network (OPSNET) is the official source of National Airspace (NAS) air traffic operations and delay data.

Problem

Capacity Uncertainty

- Lack of optimization models for **multi-airport/airspace settings**.
- **Large-scale nature** of the deterministic problem poses tractability challenge.

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This Work

- **First application** of robust and adaptive optimization to Air Traffic Flow Management (ATFM) under capacity uncertainty.
- Aims to **bridge the gap** between deterministic network models and stochastic single-airport models.

Our Proposal

- Construct **low-dimensional** uncertainty sets.
- Propose **tractable solution methodologies** for the robust and adaptive problems.
- Approach in the spirit of **Robust Optimization (RO)** rather than **Stochastic Programming**.

Robust-Adaptive Paradigm

Robust Optimization

- Construct **uncertainty set** for possible data realizations
- Optimize **worst-case** objective (while ensuring feasibility for all scenarios)
- **Tractable** for a large class of optimization problems

Adaptive Optimization

- Paradigm for **multi-period** decision-making
- Decisions are adapted to capture the **progressive information** revealed over time
- Subclass: policy-based adaptability. **Affine policies** widely used and computationally tractable

Starting Point

The decision variables are:

$$w_{j,t}^{f} = \begin{cases} 1, & \text{if flight } f \text{ arrives at sector } j \text{ by time } t, \\ 0, & \text{otherwise.} \end{cases}$$

$$s_{f,f'} = \begin{cases} 1, & \text{if there is a reversal b/w } f \text{ and } f', \\ 0, & \text{otherwise.} \end{cases}$$

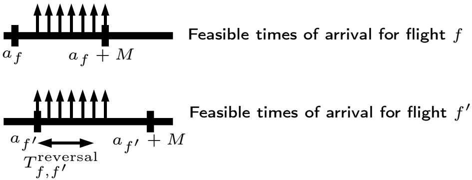

Figure: A reversible pair of flights $(f, f') \in \mathcal{R}$.

Starting Point

Concise description of the deterministic model (TFMP) [2]:

$$
\begin{aligned}
IZ_{\mathrm{TFMP}} = \min_{\mathbf{w}} \quad & \mathbf{c}'\mathbf{w} \\
\text{s.t.} \quad & \mathbf{A}\mathbf{w} \leq \mathbf{b}, \\
& \mathbf{w} \in \{0,1\}^n.
\end{aligned}
\tag{1}
$$

- TFMP: Superior integrality properties and computational efficiency.
- We introduce a uncertainty set $\mathcal{U}$ on the capacity $\mathbf{b}$.

[2] Please refer to the end for complete description of the constraints. References: 1) Bertsimas and Stock-Patterson, *Operations Research*, 1998; and 2) Bertsimas and Gupta, *Operations Research*, 2011.

Robust and Adaptive Problems

Multi-stage Adaptive problem (Π^T_{Adapt}).

$$IZ_{\text{Adapt}} = \min_{\mathbf{w}_1, \mathbf{w}_i(\mathbf{b}_{[i]})} \left[\mathbf{c}_1' \mathbf{w}_1 + \max_{\mathbf{b} \in \mathcal{U}} \left[\mathbf{c}_2' \mathbf{w}_2(\mathbf{b}_{[2]}) + \cdots + \max_{\mathbf{b} \in \mathcal{U}} \left[\mathbf{c}_{T-1}' \mathbf{w}_{T-1}(\mathbf{b}_{[T-1]}) + \max_{\mathbf{b} \in \mathcal{U}} \mathbf{c}_T' \mathbf{w}_T(\mathbf{b}_{[T]}) \right] \right] \right]$$

$$\text{s.t. } \mathbf{A}_1 \mathbf{w}_1 + \sum_{i=2}^{T} \mathbf{A}_i \mathbf{w}_i(\mathbf{b}_{[i]}) \leq \mathbf{b}, \quad \forall\, \mathbf{b} \in \mathcal{U}, \qquad (\Pi^T_{\text{Adapt}})$$

$$\mathbf{w}_i(\mathbf{b}_{[i]}) \in \{0,1\}^{n_i}.$$

Robust problem (Π_{Rob}).

$$IZ_{\text{Rob}} = \min_{\mathbf{w}} \sum_{i=1}^{T} \mathbf{c}_i \mathbf{w}_i$$

$$\text{s.t. } \sum_{i=1}^{T} \mathbf{A}_i \mathbf{w}_i \leq \mathbf{b}, \quad \forall\, \mathbf{b} \in \mathcal{U}, \qquad (\Pi_{\text{Rob}})$$

$$\mathbf{w}_i \in \{0,1\}^{n_i}.$$

Outline

Weather-front based approach

Premise:

- Correlations over space and time
- Small number of weather-fronts moving through a day

Uncertain parameters:

- Time of arrival: $T_a \in \{\underline{T}_a, \ldots, \overline{T}_a\}$.
- Duration: $d \in \{\underline{d}, \ldots, \overline{d}\}$.
- Reduction in capacity: $\alpha \in \{\underline{\alpha}, \ldots, \overline{\alpha}\}$.

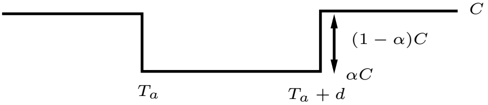

Figure: Depiction of the capacity profile under a *weather-front* based uncertainty set for a single affected airspace element.

Motivation

- Practical evidence of the applicability of fitting step functions to actual capacity profiles [3]
- Our proposal is compatible with existing available data

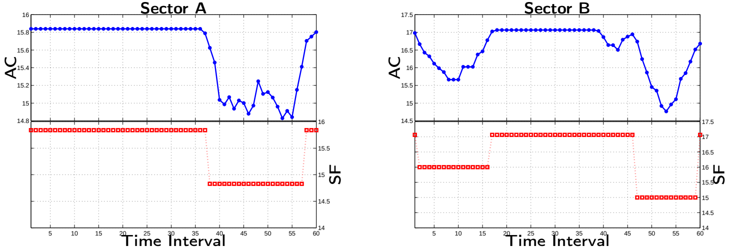

Figure: Applicability of step functions to model capacity profiles. **AC**: actual sector capacity; **SF**: step function capacity.

[3] capacity profiles for two sectors from data obtained from Lincoln Labs

Model for the airspace

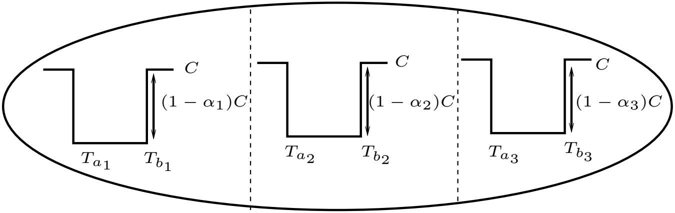

Figure: Traversal of a *weather-front* across the NAS. It has three phases over the course of its existence.

- Each affected airspace element associated with a 3-tuple (T_a, d, α).

Weather-front: Discrete Uncertainty Set

Variables

$$y_t = \begin{cases} 1, & \text{if capacity drops } \textit{by} \text{ time } t, \\ 0, & \text{otherwise.} \end{cases}$$

$$z_t = \begin{cases} 1, & \text{if capacity revives } \textit{by} \text{ time } t, \\ 0, & \text{otherwise.} \end{cases}$$

$$b_t = \underbrace{\underbrace{C(1-y_t)}_{\text{before drop}} + \mathrlap{\overbrace{}^{\text{after revival}}} \overbrace{\alpha C y_t}^{\text{before revival}}}_{\text{after drop}} \quad + (1-\alpha) C z_t$$

Weather-front: Discrete Uncertainty Set

Description for a particular realization of α:

$$\mathcal{U}_\alpha = \Big\{ \mathbf{b} \in \mathbb{Z}_+^m \,|\, b_t = C(1-y_t) + \alpha C y_t + (1-\alpha) C z_t,\ \forall t \in \{\underline{T}_a, \ldots, \overline{T}_b\};$$
$$b_t = C,\ \forall t \in \mathcal{T} \setminus \{\underline{T}_a, \ldots, \overline{T}_b\};$$
$$y_t \le y_{t+1};\ z_t \le z_{t+1};\ z_t \le y_t;$$
$$y_{\overline{T}_a} = 1;\ z_{\underline{T}_b - 1} = 0;\ z_{\overline{T}_b} = 1;\ y_t, z_t \in \{0,1\} \Big\}$$

Overall uncertainty set for the capacity profile:

$$\mathcal{U} = \left(\bigcup_{\alpha \in \{\underline{\alpha}, \ldots, \overline{\alpha}\}} \mathcal{U}_\alpha \right)$$

Weather-front: Polyhedral Description

- The optimization problem under $\mathcal{U}$ (discrete set) is equivalent to optimization problem under $\mathrm{conv}(\mathcal{U})$.
- **Challenge:** A polyhedral description of $\mathrm{conv}(\mathcal{U})$?

A polyhedral description of conv($\mathcal{U}_\alpha$).

$$\begin{aligned}\mathcal{P}_\alpha = \Big\{\mathbf{b} \in \mathbb{R}^m_+ |\ & b_t = C(1-y_t) + \alpha C y_t + (1-\alpha) C z_t,\ \forall t \in \{\underline{T}_a, \ldots, \overline{T}_b\}; \\ & b_t = C,\ \forall t \in \mathcal{T} \setminus \{\underline{T}_a, \ldots, \overline{T}_b\}; \\ & y_t \le y_{t+1};\ z_t \le z_{t+1};\ z_t \le y_t; \\ & y_{\overline{T}_a} = 1;\ z_{\underline{T}_b - 1} = 0;\ z_{\overline{T}_b} = 1;\ 0 \le y_t, z_t \le 1\Big\}\end{aligned}$$

Weather-front: Polyhedral Description

Theorem. $\mathcal{P}_\alpha$ is integral and is exactly the convex hull of $\mathcal{U}_\alpha$.

$$\mathcal{U}_\alpha \subseteq \mathrm{conv}(\mathcal{U}_\alpha) = \mathcal{P}_\alpha$$

Proof. Uses the technique of randomized rounding.

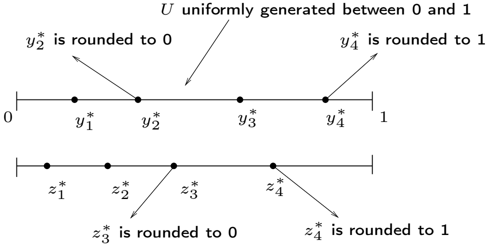

Figure: Illustration of the geometry of the *randomized rounding* algorithm for proving the integrality of polyhedron $\mathcal{P}_\alpha$.

Weather-front: Polyhedral Description

Theorem. $\mathcal{P}_{\underline{\alpha}}$ is exactly $\operatorname{conv}(\mathcal{U})$, i.e.,

$$\operatorname{conv}(\mathcal{U}) = \mathcal{P}_{\underline{\alpha}}$$

- $\Longrightarrow$ uncertainty governing α not required (input needed is only $\underline{\alpha}$).
- $(T_a, d, \alpha) \longrightarrow (T_a, d)$ for each airspace element.
- **Comparison with Hypercube Uncertainty Set.** 3 WFs (3 phases; 3 affected airspace elements, $|\overline{T}_a - \underline{T}_a + 1| = 3$, $|\overline{d} - \underline{d} + 1| = 3$ and $d \geq 3$). Hypercube: 162 parameters, WF: 54!

Outline

Solving the Robust Problem

Definition. $\mathbf{b}_{\min} = (\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_m)$, where $\underline{b}_i = \min \{ b_i | \, \mathbf{b} \in \mathcal{U} \}$.

Theorem. For an arbitrary uncertainty set $\mathcal{U}$, the robust TFMP problem is equivalent to solving the following modified TFMP instance:

$$\begin{aligned} IZ_{\text{TWURob}} = \min_{\mathbf{w}} \quad & \mathbf{c}'\mathbf{w} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{w} \leq \mathbf{b}_{\min}, \\ & \mathbf{w} \in \{0,1\}^n. \end{aligned} \tag{2}$$

Solving the Robust Problem

Definition. $\mathbf{b}_{\min} = (\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_m)$, where $\underline{b}_i = \min \left\{ b_i | \ \mathbf{b} \in \mathcal{U} \right\}$.

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& \mathbf{w} \in \{0,1\}^n.
\end{aligned}
\tag{2}
$$

- Solving robust instance is equivalent to solving another TFMP (deterministic) instance ($\Longrightarrow$ retains the attractive integrality properties and computational efficiency of TFMP model).

Intuition for solving the Multi-stage Adaptive Problem.

Procedure.

- Construct a **polyhedral description** of the uncertainty set
- Divide the uncertainty set into **multiple simplices**
- An **affine policy** is optimal for each piece

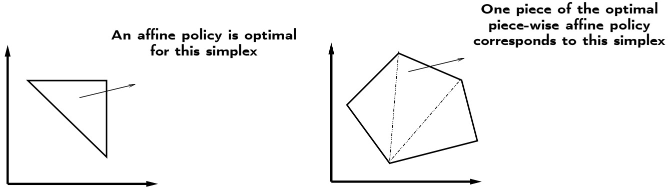

Figure: **Left:** A simplex uncertainty set in $\mathbb{R}^2$. **Right:** A polyhedral set in $\mathbb{R}^2$ broken into three simplices.

Weather-front: # of Extreme Points

Exact Expression.

$$\mathcal{E} = \Pi_{i=1}^{k}\Pi_{j=1}^{p_i}\Big((\overline{T}_{a_i} - \underline{T}_{a_i} + 1) \times (\overline{d}_i - \underline{d}_i + 1)\Big)^{|\mathcal{W}_i^j(\mathcal{S})| + |\mathcal{W}_i^j(\mathcal{K})|}$$

Theorem.
Let k be the number of weather fronts and P be the maximum number of phases across all fronts. Let,

$$\tau = \max_i \left\{\overline{T}_{a_i} - \underline{T}_{a_i} + 1,\ \overline{d}_i - \underline{d}_i + 1\right\} \qquad \Delta = \max_{i,j}\left\{|\mathcal{W}_i^j(\mathcal{S})|,\ |\mathcal{W}_i^j(\mathcal{K})|\right\}$$

Then, $\mathcal{E}$ can be upper bounded as follows:

$$\mathcal{E} \leq \tau^{4k*\Delta*P}$$

Weather-front: # of Extreme Points

τ	Δ	Upper Bound on $\mathcal{E}$ $(k = 1)$	
		$P = 1$	$P = 2$
1	1	**1**	**1**
2	2	**256**	**65536**
2	3	**4096**	16777216
3	2	**6561**	43046721
3	3	531441	$2.8*10^{11}$

Table: The bold numbers indicate the attractive cases from the point of view of obtaining a single affine policy.

Outline

Performance of Robust Counterpart

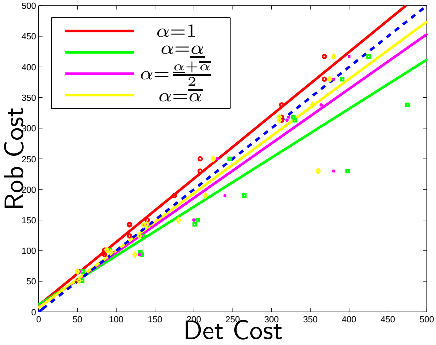

Figure: *Utility of Robust Solutions.*

- the robust schedules come at a small cost
- the cost of the protracted deterministic solution is generally higher

Performance of Robust Counterpart

Region (# of Flights)	$\mathbf{Z}_{\text{Det}}$	Sol. Time (sec.)	$\mathbf{Z}_{\text{Rob}}$	Sol. Time (sec.)	% Nonint	Uncertainty Set (# of EPs)
North-East (500-1000)	208	378	229	407	0	81
	227	515	230	643	0	256
	145	636	161.5	638	0.15	729
	281	466	281	533	0	14677618
	935	4839	971.5	4721	0	20736000
Central (500-1000)	650	1996	677	1984	0	81
	193	103	193	95	0	256
	647	1770	675	1779	0	729
	935	4685	1013	4607	0	4299816
	240	401	248	402	0.10	1679616

Table: Computational Experience with TWURob.

Performance of Adaptive Counterpart

Region (# of Flights)	$\mathbf{Z}_{\mathrm{Rob}}$	Sol. Time (sec.)	$\mathbf{Z}_{\mathrm{Adapt}}$	Sol. Time (sec.)	% Nonint	Uncertainty Set (# of EPs)
North-East (100-150)	92	84	92	265	0	81
	29.5	73	29.5	491	0	128
	54	55	54	369	0	256
	113	127	112	326	0	4096
Central (100-150)	27	43	27	813	0	1696
	22	34	22	287	0	225
	86	98	86	257	0	729
	35	63	35	382	0	1728

Table: Computational Experience with TWUAdapt.

Outline

Conclusions

Summary

- An effort to address **stochasticity in a network ATFM setting**.
- **Low-dimensional**, realistic model of uncertainty set.
- **Tractable solution methodologies** for the robust and adaptive problem.

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- An effort to address **stochasticity in a network ATFM setting**.
- **Low-dimensional**, realistic model of uncertainty set.
- **Tractable solution methodologies** for the robust and adaptive problem.

Takeaways

- Robust problem equivalent to solving another **deterministic instance**.
- **Affine policies are optimal** for the LP relaxation of the adaptive equivalent.

Questions?

Thank You!

BackUp Slides

Air Traffic Flow Management (ATFM)

What is ATFM?

- Set of **strategic processes** that reduce congestion costs.
- **Ground Delay Programs (GDPs)** - one of the major ATFM tools.
- **Airspace Flow Programs (AFPs)** - a more recently used tool; others include assigning air-borne delays, dynamic re-routing and speed control.

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Element of Stochasticity

- Only handled reasonably in the context of GDPs.
- Multi-airport/Airspace settings not really addressed.

Literature: Network Models on ATFM

Two Paradigms

- **Lagrangian**: trajectory-based.
- **Eulerian**: control of aggregate flow.

Goal

- Minimize the **total delay costs** across all flights.
- Focus is on maximizing **system throughput**.

Limitations

- **Deterministic models**.

Literature: Stochastic SAGHP

Papers

- **Ball, Hoffman, Odoni and Rifkin** (2003) - A Stochastic IP with Dual Network Structure.
- **Kotynek, Richetta** (2006) - Relating Equity and Integrality.
- **Mukherjee, Hansen** (2007) - A Dynamic Stochastic Model.

Starting Point: Bertsimas Stock-Patterson Model (1998)

Decision Variables

$$w_{j,t}^{f} = \begin{cases} 1, & \text{if flight } f \text{ arrives at sector } j \text{ by time } t, \\ 0, & \text{otherwise.} \end{cases}$$

- Definition with "*by*" rather than "*at*" critical for models' excellent performance.

Objective Function

$$IZ_{\mathrm{TFMP}} = \min_{\mathbf{w},\mathbf{s}} \sum_{f \in \mathcal{F}} \left(\delta \cdot \Big(\sum_{t \in T^f_{\mathrm{dest}_f}} t \cdot (w^f_{\mathrm{dest}_f,t} - w^f_{\mathrm{dest}_f,t-1}) - a_f \Big) + (1-\delta) \right.$$

$$\left. \Big(\sum_{t \in T^f_{\mathrm{orig}_f}} t \cdot (w^f_{\mathrm{orig}_f,t} - w^f_{\mathrm{orig}_f,t-1}) - d_f \Big) \right) + \lambda \cdot \left(\sum_{(f,f') \in \mathcal{R}} s_{f,f'} \right)$$

(TFMP) Recap

Capacity Constraints

$$\sum_{f \in \mathcal{F}:\text{orig}_f = k} (w^f_{k,t} - w^f_{k,t-1}) \leq D_k(t), \qquad \forall k \in \mathcal{K}, \;\; t \in \mathcal{T}, \tag{3}$$

$$\sum_{f \in \mathcal{F}:\text{dest}_f = k} (w^f_{k,t} - w^f_{k,t-1}) \leq A_k(t), \qquad \forall k \in \mathcal{K}, \;\; t \in \mathcal{T}, \tag{4}$$

$$\sum_{f \in \mathcal{F}: j \in \mathcal{S}_f, j' = L^f_j} (w^f_{j,t} - w^f_{j',t}) \leq S_j(t), \qquad \forall j \in \mathcal{S}, \;\; t \in \mathcal{T}, \tag{5}$$

- Constraints satisfying the departure, arrival and sector capacities respectively.

(TFMP) Recap

Connectivity Constraints

$$w^f_{j,t} - w^f_{j',t-l_{fj'}} \leq 0, \qquad \forall f \in \mathcal{F},\ \ t \in T^f_j,\ \ j \in \mathcal{S}^f \ \ : j' = P^f_j, \tag{6}$$

$$w^f_{\text{orig}_f,t} - w^{f'}_{\text{dest}_{f'},t-s_f} \leq 0, \qquad \forall (f,f') \in \mathcal{C},\ \ \forall t \in T^f_k, \tag{7}$$

$$w^f_{j,t-1} - w^f_{j,t} \leq 0, \qquad \forall f \in \mathcal{F},\ \ j \in \mathcal{S}^f,\ \ t \in T^f_j, \tag{8}$$

- Constraints satisfying sector, flight and time connectivity respectively.
- Connectivity constraints are facet-defining which is why this model performs very well computationally.

(TFMP) Recap

Fairness Constraints

$$w^{f'}_{\text{dest}_{f'},t} \leq w^{f}_{\text{dest}_{f},t} + s_{f,f'}, \qquad \forall (f,f') \in \mathcal{R}, \ \ t \in T^{\text{reversal}}_{f,f'}, \qquad (9)$$

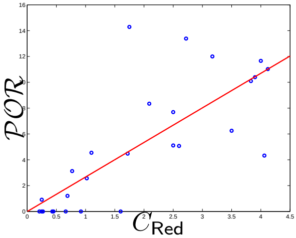
$$w^{f}_{\text{dest}_{f},t} \leq w^{f'}_{\text{dest}_{f'},t} + 1 - s_{f,f'}, \qquad \forall (f,f') \in \mathcal{R}, \ \ t \in T^{\text{reversal}}_{f,f'}, \qquad (10)$$

- Constraints model a reversal.
- Fairness corresponds to controlling the number of reversals.

Data and Setup

- Airspace: 10 by 10 grid (100 sectors). 55 major US airports mapped to one of these 100 sectors.
- Different regions of the airspace subjected to simulated weather-fronts. (T_a, d, α) generated randomly from appropriate intervals.
- Solution Tool: *Robust Optimization Made Easy* (ROME).

Characteristics of Robust Solutions



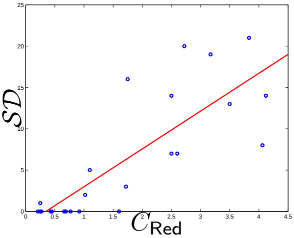


Figure: **Left:** Price of robustness ($\mathcal{POR}$) as a function of capacity reduction. **Right:** Schedule deviation ($\mathcal{SD}$) as a function of capacity reduction.

- Approximately **linear relationship** of $\mathcal{POR}$ and $\mathcal{SD}$ with C_{Red} (the red line corresponds to the best linear fit.)

Relating the Robust and Adaptive problems

Proposition

Consider the multi-stage adaptive optimization problem Π^T_{Adapt} *and its robust counterpart* Π_{Rob}*. If* $\mathbf{b}_{\min} \in \mathcal{U}$*, then,*

$$IZ_{\text{Adapt}} = IZ_{\text{Rob}}.$$

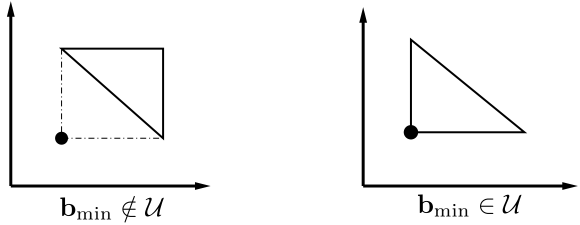


Figure: Example uncertainty sets with and without $\mathbf{b}_{\min}$ (black filled circle denotes $\mathbf{b}_{\min}$).

Toolbox to solve the Adaptive Problem

Theorem

Consider the problem $\Pi^T_{\text{Adapt}}(\mathcal{U})$ *such that* $\mathcal{U}$ *is a simplex. Then, there is an optimal multi-stage solution* $\hat{\mathbf{w}}_i(\mathbf{b})$ *such that* $\hat{\mathbf{w}}_i(\mathbf{b})$ *are affine functions of* $\mathbf{b}$*, i.e., for all* $\mathbf{b} \in \mathcal{U}$*,*

$$\hat{\mathbf{w}}_i(\mathbf{b}) = \mathbf{P}_i\mathbf{b} + \mathbf{q}_i, \tag{11}$$

where $\mathbf{P}_i \in \mathbb{R}^{n_i \times m}$*,* $\mathbf{q}_i \in \mathbb{R}^{n_i}$*.*

Weather-front: # of Extreme Points

Description (Denoted By)	Uncertainty Set	# of Extreme Points
1 **WF**, **Single airspace element, one phase** (EP_i)	$\mathcal{P}_{\underline{\alpha}_i} = \{ \mathbf{b} \in \mathbb{R}^m_+ \mid b_t = C(1-y_t) + \underline{\alpha}_i C y_t + (1-\underline{\alpha}_i) C z_t,\ \forall t \in \{\underline{T}_{a_i}, \ldots, \overline{T}_{b_i}\};\ b_t = C,\ \forall t \in \mathcal{T} \setminus \{\underline{T}_{a_i}, \ldots, \overline{T}_{b_i}\};\ y_t \le y_{t+1};\ z_t \le z_{t+1};\ z_t \le y_t;\ 0 \le y_t, z_t \le 1 \}$	$(\overline{T}_{a_i} - \underline{T}_{a_i} + 1) \times (\overline{d}_i - \underline{d}_i + 1)$
1 **WF**, **Airspace, one phase** $(\mathcal{E}_i^j)$	$\mathcal{P}_{\underline{\alpha}_i}^{\mathcal{AS},\mathcal{OP}} = \bigoplus_{k \in \mathcal{A}} \mathcal{P}_{\underline{\alpha}_i}^k$	$EP_i^{\lvert \mathcal{W}_i^j(\mathcal{S}) \rvert + \lvert \mathcal{W}_i^j(\mathcal{K}) \rvert}$
1 **WF**, **Airspace, all phases** $(\mathcal{E}_i)$	$\mathcal{P}_{\underline{\alpha}_i}^{\mathcal{AS},\mathcal{AP}} = \bigoplus_{j=1}^{p_i} \mathcal{P}_{\underline{\alpha}_i}^j$	$\Pi_{j=1}^{p_i} \mathcal{E}_i^j$
k **WF**, **Airspace, all phases** $(\mathcal{E})$	$\mathcal{P}_{\underline{\alpha}}^{\mathcal{AS},\mathcal{AP}} = \bigoplus_{i=1}^{k} \mathcal{P}_{\underline{\alpha}_i}$	$\Pi_{i=1}^{k} \mathcal{E}_i$

Table: Number of extreme points for the weather-front based polytope.

$\mathcal{AS}$ denotes airspace, $\mathcal{OP}$ denotes one-phase and $\mathcal{AP}$ denotes all-phases.

Optimization Under Discrete Uncertainty Set

OptiDU denotes optimization problem under discrete uncertainty set ($\mathcal{U}_0$: discrete uncertainty set).

$$\begin{aligned} Z_{\text{OptiDU}} = \min_{\mathbf{x}} \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \;\; \forall \mathbf{b} \in \mathcal{U}_{\mathbf{0}}. \end{aligned} \tag{12}$$

- Work with $\text{conv}(\mathcal{U}_0)$ (convex hull of $\mathcal{U}_0$) to enable a polyhedral description.
- The optimization problem under $\mathcal{U}_0$ (discrete set) is equivalent to optimization problem under $\text{conv}(\mathcal{U}_0)$.

Notation.

The model's formulation requires definition of the following notation:

$$\mathcal{K} \equiv \text{set of airports,}$$

$$\mathcal{W} \equiv \text{set of airlines,}$$

$$\mathcal{W}^k \equiv \text{set of flights belonging to airline } k\text{,}$$

$$\mathcal{S} \equiv \text{set of sectors,}$$

$$\mathcal{S}^f \subseteq \mathcal{S} \equiv \text{set of sectors that can be flown by flight } f\text{,}$$

$$\mathcal{F} \equiv \text{set of flights,}$$

$$\mathcal{T} \equiv \text{set of time periods,}$$

$$\mathcal{C} \equiv \text{set of pairs of flights that are continued,}$$

$$\mathcal{R} \equiv \text{set of pairs of flights that are reversible, (see definition below)}$$

$$\mathcal{P}_i^f \equiv \text{set of sector } i\text{'s preceding sectors,}$$

$$\mathcal{L}_i^f \equiv \text{set of sector } i\text{'s subsequent sectors,}$$

Notation (Continued).

$$
\begin{aligned}
D_k(t) &\equiv \text{departure capacity of airport } k \text{ at time } t, \\
A_k(t) &\equiv \text{arrival capacity of airport } k \text{ at time } t, \\
S_j(t) &\equiv \text{capacity of sector } j \text{ at time } t, \\
d_f &\equiv \text{scheduled departure time of flight } f, \\
a_f &\equiv \text{scheduled arrival time of flight } f, \\
s_f &\equiv \text{turnaround time of an airplane after flight } f, \\
orig_f &\equiv \text{airport of departure of flight } f, \\
dest_f &\equiv \text{airport of arrival of flight } f, \\
l_{fj} &\equiv \text{minimum number of time units that flight } f \text{ must spend in sector } j, \\
D &\equiv \text{maximum permissible delay for a flight}, \\
T_j^f = [\underline{T}_j^f, \overline{T}_j^f] &\equiv \text{set of feasible time periods for flight } f \text{ to arrive in sector } j, \\
\underline{T}_j^f &\equiv \text{first time period in the set } T_j^f, \\
\overline{T}_j^f &\equiv \text{last time period in the set } T_j^f.
\end{aligned}
\qquad (13)
$$