Centre for Transport Studies
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Static Multi-Class Traffic Equilibria under Congestion Pricing

P O Lindberg, Department of Transport and Economics, KTH, Stockholm

Synopsis

We will look at effects of link tolls in static traffic networks having user classes with different time values.

These results, which are somewhat peculiar, have been presented earlier, but will be summarized here.

In particular we study ***Fixed*** tolls and (flow dependent) ***Marginal Social Cost*** (MSC) tolls.

Both the Fixed toll and the MSC toll equilibrium problems can be stated in non-symmetric as well as symmetric forms, the latter corresponding to optimization problems.

The Fixed toll equilibrium problem is a convex optimization problem, with unique total link flows if the link travel times are strictly increasing. The class flows need not be unique though. But the ***total value of travel time*** (a natural societal objective) is unique.

The MSC equilibrium problem is in general non-convex, and may thus have several local minima (and equilibria).

Implementing equilibrium MSC tolls as fixed tolls leads to fixed toll equilibria with equilibrium sets containing the MSC equilibria, but maybe also other equilibria.

Finally we will quickly look at how to determine fixed tolls for a subset of the links, so as to minimize total value of travel time.

Setup

As said, we study *static*, *multi-class traffic equilibria*, corresponding to a classical Traffic Assignment setting.

This may seem unnecessary limited. But the situation is complex enough. And adding more reality, will give more complexity.

Thus, we assume that we have *user classes* $k \in K$, only differing in their *time values* (VoT), v_k.

We further have a *network* consisting of *nodes* $n \in N$ and directed *links* (arcs) $a \in A$, connecting the nodes.

Let $W \subseteq N \times N$ be the set of *OD (origin-destination) pairs* (i.e. where traffic originates and ends).

For each OD-pair $w = (o_w, d_w)$ there is, for each class *k*, a fixed *traffic demand* q_w^k (e.g. in veh./h) from o_w to d_w.

Equilibrium traffic is assumed to take least costly routes from origin to destination (Wardrop condition).

The traffic give rise to *link flows* f_a^k, of class *k* in link *a*.

We further assume the link *travel time*, $t_a(f_a^{tot})$, only depends on the *total link flow* $f_a^{tot} = \sum_k f_a^k$ on the link.

Fixed tolls

Even though MSC tolls depend on the link flows, they typically have to be implemented as *fixed tolls*.

Fixed tolls also are interest in them selves.

Thus we first look at multi-class equilibria under fixed tolls. Let p_a be the toll in link *a*.

Thus we define the *generalized cost* $\bar{c}_a^k$ and the *generalized time* $\bar{t}_a^k$ of link *a* for class *k* respectively as

$\bar{c}_a^k(f_a) = v_k t_a\left(f_a^{tot}\right) + p_a$, and

$\bar{t}_a^k(f_a) = t_a\left(f_a^{tot}\right) + p_a / v_k$.

Switching between $\bar{c}_a^k$ and $\bar{t}_a^k$, one just scales all link and route costs for a given user class with the same scalar (v_k or $1/v_k$). This does not change the equilibria, since equilibrium is scale invariant in the cost.

Thus we define the *fixed toll multi-class user equilibrium problem* (with class specific time values), as the problem to find an equilibrium with costs c_a^k equal to $\bar{c}_a^k$ or equivalently to $\bar{t}_a^k$.

It is well-known that if he costs are symmetric (in our case if $\dfrac{\partial \bar{c}_a^k(f_a)}{\partial f_a^l} = \dfrac{\partial \bar{c}_a^l(f_a)}{\partial f_a^k}$)

then the equilibrium can be stated as an optimization problem, which is beneficial.

Checking symmetry, we see, (using $\partial f_a^{tot} / \partial f_a^k = 1$), that the fixed toll equilibrium problem with costs

$\bar{c}_a^k$ is not symmetric, whereas the one expressed in $\bar{t}_a^k$ is.

The optimization for latter problem has the objective $\bar{I}(f) = \sum_{a \in A}\left[\int_0^{f_a^{tot}} t_a(u)du + p_a \sum_{k \in K} f_a^k / v_k\right]$.

Ambiguity

If the link times t_a are strictly increasing (which is the usual case). Then the objective

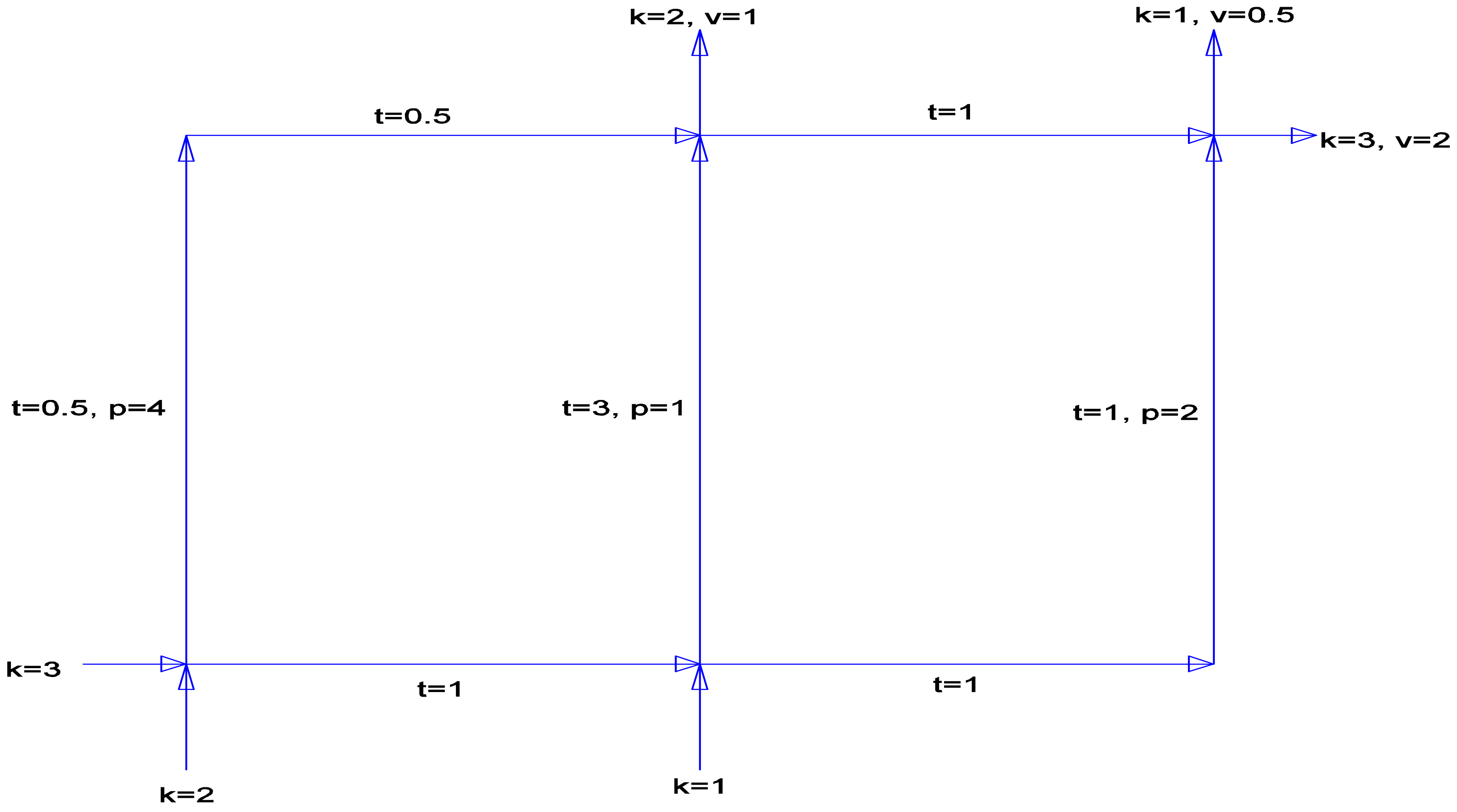
$$\bar{I}(f) = \sum_{a \in A} \left[\int_0^{f_a^{tot}} t_a(u) du + \sum_{k \in K} f_a^k p_a / v_k \right]$$, is strictly convex in the total link flows. Then the total link flows

are unique at the optimum (i.e. equilibrium).

This implies that the link times are unique too, and hence also the generalized link cost and link times, as well as generalized cost and times for all routes.

But the solution need not be unique in the terms of the class specific link flows, for instance if there are two routes with the same sum of tolls and same sum of link-times.

But the ambiguity can be more complex as can be seen in the following network.



Ambiguous network

However, the situation is still well-behaved. Under the assumption of strictly increasing link travel times it turns out that the *total perceived value of travel time*

$$V(f) = \sum_a \sum_k v_k t_a(f_a^{tot}) f_a^k$$

is *unique* over all fixed toll equilibria.

Further, under strictly increasing link travel times, the total equilibrium link flows $f_a^{tot}(p)$, can be shown to be continuous as functions of the tolls *p*. This is important for the implementation of (flow dependent) MSC tolls.

These results show that one could, at least in principle, determine tolls e.g. on a subset of the links, so as to minimize the total perceived value of travel time.

Marginal Social Cost (MSC) *tolls*

A marginal user in link *a* inflicts the delay $t'_a(f_a^{tot})$ on other users. The total value of this delay is $t'_a\left(f_a^{tot}\right)\sum_{m\in K} v_m f_a^m$. To internalize this external cost, it is thus natural to use flow dependent MSC tolls $p_a = p_a(f) = t'_a\left(f_a^{tot}\right)\sum_{m\in K} v_m f_a^m$. Using these tolls in our link costs (and times), we get the *MSC link costs*

$$\tilde{c}_a^k(f_a) = v_k t_a\left(f_a^{tot}\right) + t'_a\left(f_a^{tot}\right)\sum_{m\in K} v_m f_a^m,$$

and the *MSC link times*

$$\tilde{t}_a^k(f_a) = t_a\left(f_a^{tot}\right) + t'_a\left(f_a^{tot}\right)\sum_{m\in K} v_m f_a^m / v_k.$$

Using these as costs in the equilibrium definition, we get *MSC equilibria*.

As for fixed tolls, we can check symmetry, and we find that, in this case the $\tilde{t}_a^k$ are asymmetric, whereas the $\tilde{c}_a^k$ are symmetric. The objective corresponding to the $\tilde{c}_a^k$ turns out to be

$$\tilde{I}(f) = \sum_{a\in A} t_a\left(f_a^{tot}\right)\sum_{k\in K} v_k f_a^k,$$

i.e. $\tilde{I}(f) = V(f)$ the *total perceived value of travel time* in the network.

The MSC equilibria are flows, where $V(f)$ has no feasible descent directions. These could be local minima, saddle points or other stationary points of $V(f)$. They could even be local maxima!

Implementation of MSC equilibria

As said above, MSC equilibria have to be implemented as fixed-toll equilibria.

Recall that the MSC toll in link *a* is $p_a = p_a(f) = t'_a\left(f_a^{tot}\right)\sum_{m\in K} v_m f_a^m$.

It is fairly elementary that if $\hat{f}$ is the MSC flow equilibrium, then it is also a fixed-toll equilibrium for the fixed toll $\hat{p}_a = p_a(\hat{f}) = t'_a\left(\hat{f}_a^{tot}\right)\sum_{m\in K} v_m \hat{f}_a^m$.

But, as you recall, implementing fixed tolls, does not necessarily give back the computed flows, but we get at least a flow with the same value of the total perceived value of travel time.

A two-link Example

We now consider a small example that is instrumental in showing that the MSC objective $V(f)$ in general is non-convex.

Consider a network consisting of a single OD pair w connected by two links a and b with identical travel time functions $t_a(u) = t_b(u) = u$ (Figure 1). Assume there are two user classes with time values $v_1 = 1$ and $v_2 = 5$, respectively, and each with travel demand $q_w^1 = q_w^2 = 100$.

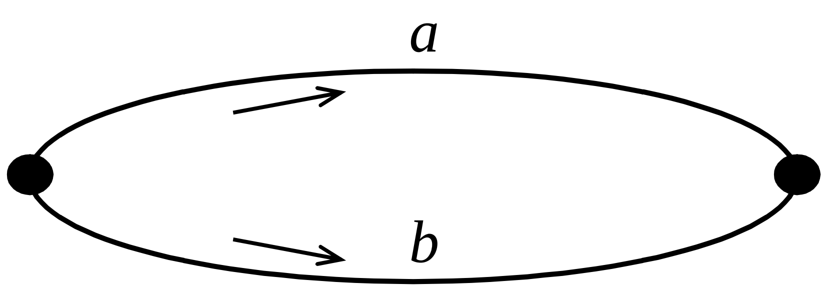


Figure 1. Example network

The feasible set *F* can be described using the *basic variables* $f_a = (f_a^1, f_a^2)$, since $f_b^k = 100 - f_a^k$.

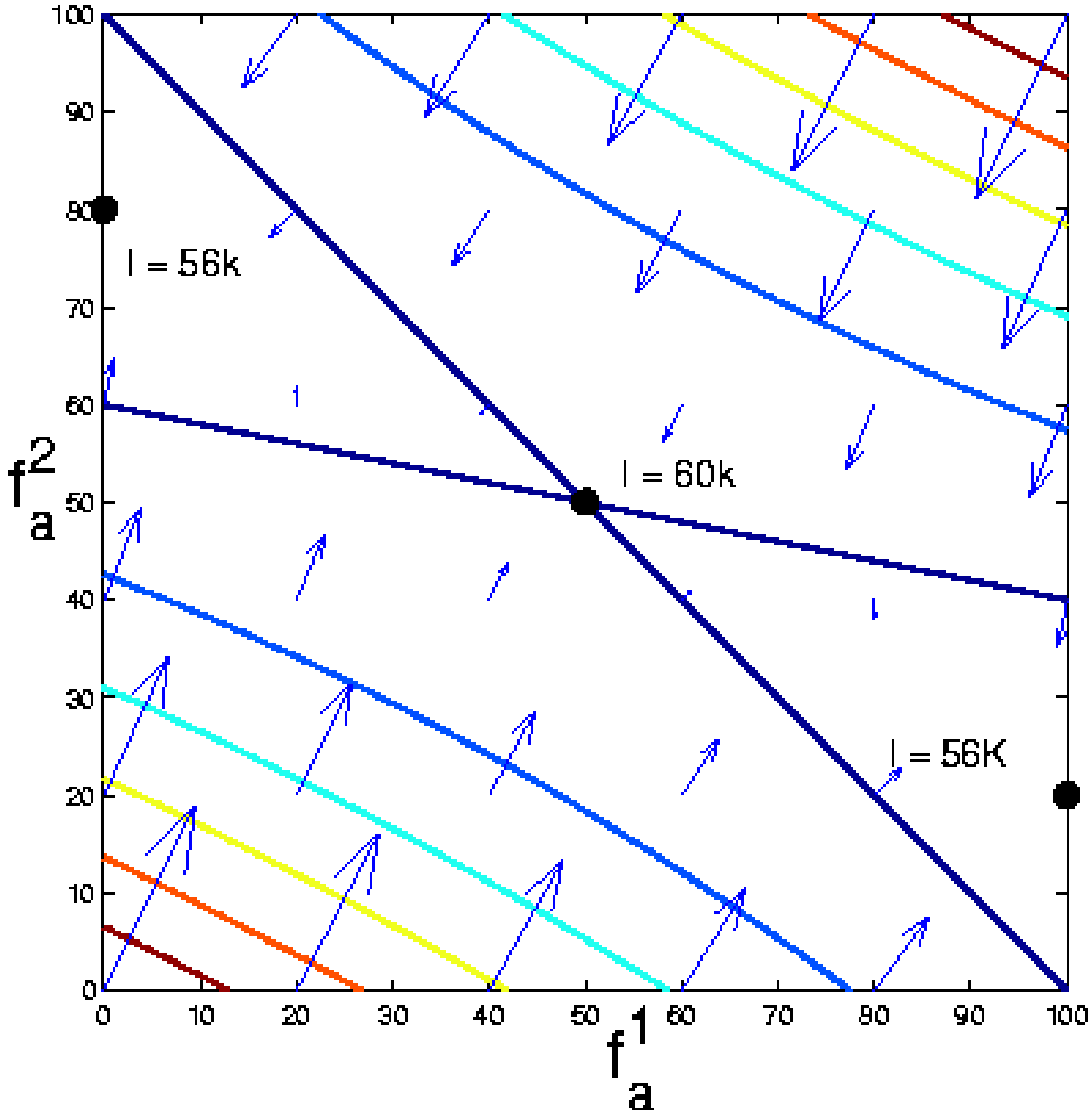
Introduction of MSC pricing leads to the MSC objective

$$V(f) = \sum_{a \in A} t_a(f_a^{tot}) \sum_{k \in K} v_k f_a^k = (f_a^1 + f_a^2)(f_a^1 + 5 f_a^2) + (f_b^1 + f_b^2)(f_b^1 + 5 f_b^2),$$

or, in terms of the basic variables

$$V(f) = (f_a^1 + f_a^2)(f_a^1 + 5 f_a^2) + (200 - f_a^1 - f_a^2)(600 - f_a^1 - 5 f_a^2).$$

In the Figure below we display level curves and negative gradient directions of *V* as functions of the basic variables.

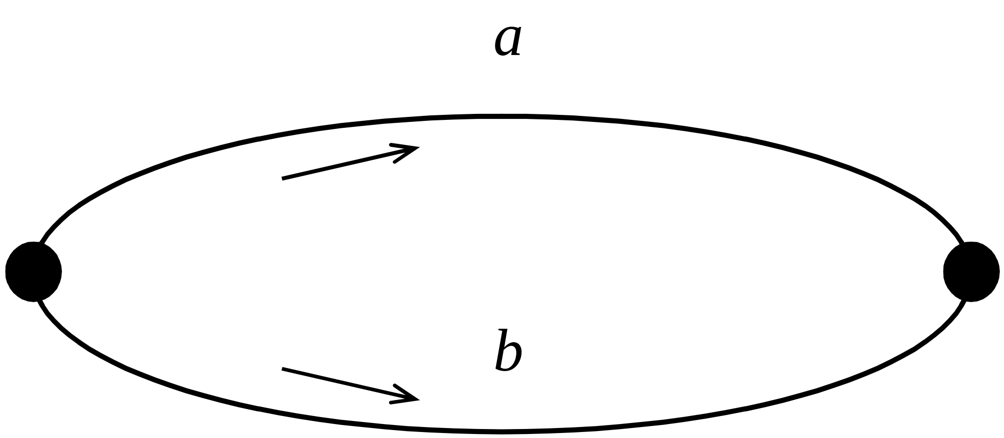


I = 56k
I = 60k
I = 56K
f_a^1
f_a^2

When the tolls $p^{(1)} = p(f^{(1)})$ or $p^{(2)}$ are enforced as fixed tolls, the only existing user equilibria are $f_a^{(1)}$ and $f_a^{(2)}$ respectively.

Implementation of the tolls $p^{(3)}$, however, does not affect the route choice, whence there is the same set of equilibria $\hat{F}$ as in the situation without tolls. Thus an equilibrium flow pattern with fixed equilibrium tolls $p^{(3)}$ need not coincide with $f^{(3)}$. However, in line with above, these flow patterns are equivalent both from the individual and the social points of view, since total flow and travel time along each link, and the total perceived value of travel time and toll revenues at any point in $\hat{F}$ is the same as at $f^{(3)}$.

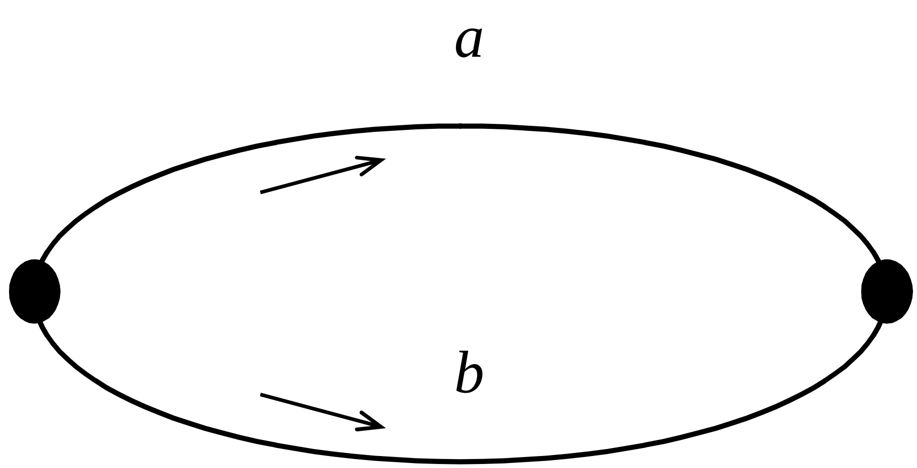
This example may seem very specialized, but the two links could represent two paths between two nodes in a network with at least two user classes. In this form it will be instrumental in proving that $V(f)$ is in general not convex



A first quick look at computing optimal fixed tolls

Ambiguity

First note that the ambiguous case, with two routes having the same travel times and toll sums is not stable. Let us modify the recent example to have a toll of 1 say in both links.



Then the flow will be 100 in both links, and both links will have travel time 100 and toll 1, so both classes are indifferent between the links. And any mix of class flows on the links is feasible.

However, adjusting the toll in link *a* to $1+dp$ changes the picture completely.

If now class 1 is indifferent between the links, then class 2, cannot be, and vice versa.

Thus we end up with $100-dp/10$ of class 1 using link *a*, and the rest link *b*.

Class 2 uses only link *b*.

Thus, we will generically, under toll changes, not have the initial setting of this example.

General case

Suppose we change the toll p_a in link *a*, from $\bar{p}_a$ to $\bar{p}_a + dp_a$ and that the corresponding link flows change from $\bar{f}(\bar{p})$ to $\bar{f}(\bar{p}+dp)$

Two routes that are equivalent for a given class *k* (before change), create a cycle for which the (generalized) link costs sum up to zero (if counted with minus sign for links against the direction of the cycle).

Let $\{g_i\}_{i\in I}$ be a cycle-basis of the total set of cycles which have zero costs for some class. Let g_i be the link- incidence vector for cycle *i*, with a +1 for links along the cycle direction, and a -1 for links against.

Let *G* be the matrix with columns g_i.

By the stability argument above, we may assume that each cycle g_i with nonzero toll has zero cost only for one class k_i. Let v_i be the VoT for that class.

Let $T(f)$ be the vector of link-travel times, as a function of the link flows.

When p_a is changed to $p_a + dp_a$, the class flows will adjust to keep the costs of the cycles at 0.

They will then adjust by decreasing flow on the route with increased toll, and vice versa. This corresponds to sending flow along the cycle. Let $d\gamma$ be the vector of total flow change along the cycles.

The vector of link-flows is $\bar{f} = \bar{f}(p)$, before the toll-change, and hence $f = \bar{f} + G\gamma$ after toll-change.

Before the toll-change, the equilibrium condition for a tolled cycle is

$$g_i^T(T(\bar{f}) + p/v_i) = 0.$$

For a non-tolled cycle it is $g_i^T T(\bar{f}) = 0$, which is to the same, since $g_i^T p = 0$. Any v_i is OK for this g_i.

After the toll-change the equilibrium condition becomes, using e_a to denote the unit vector with a +1 at the position *a*, and developing to first order:

$$0 = g_i^T(\nabla T(\bar{f} + Gd\gamma) + (p + e_a dp_a)/v_i \approx g_i^T(T(\bar{f}) + p/v_i) + g_i^T(\nabla T(\bar{f})Gd\gamma + e_a dp_a/v_i)$$

$$= \{\text{first term } 0 \text{ by eq. cond.}\} = g_i^T \nabla T(\bar{f})Gd\gamma + g_i^T e_a dp_a/v_i$$

Let $D = Diag(1/v_i)$. Then g_i^T/v_i is the *i*-th row of DG^T.

Thus assembling the g_i^T into G^T, and the g_i^T/v_i into DG^T, the equilibrium condition after toll-change becomes

$$G^T \nabla T(\bar{f})Gd\gamma = -DG^T e_a dp_a$$

And finally we get

$d\gamma = -(G^T \nabla T(\bar{f})G)^{-1} DG^T e_a dp_a$. (Note that $\nabla T(\bar{f})$ is diagonal.)

This tells how γ changes with *p* when p_a is changed.

Tracing the changes in the OD class flows we also get how the objective $V(\bar{f}(p))$ changes with p_a.

Thus we can get $\dfrac{\partial V(\bar{f}(p))}{\partial p_a}$.

Repeating for all *a* of interest we get $\nabla_p V(\bar{f}(p))$.

This gradient-info can be used e.g. for quasi-Newton methods.

Doing line-search, we must however check for depletion of cycles, or new cycles opening up.

The End