Stochastic Target

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I have benefited the collaboration of many people including: Erdinç Akyıldırım, Albert Altarovici, Yan Dolinsky, Romuald Elie, Selim Gökay, Ludovic Moreau, Johannes Muhle-Karbe, Marcel Nutz, Dylan Possamaï, Max Reppen, Jianfeng Zhang and

Nizar Touzi

Bruno Bouchard
In the classical Markovian optimal control theory, one tries to maximize (or minimize) an objective functional of the form

$$\mathbb{E} \left[ \int_0^T f(X_t^\alpha, \alpha_t)dt + \Phi(X_T^\alpha) \right]$$

over all admissible controls $\alpha$. Here $X^\alpha$ is the controlled state process.
For the stochastic target problems, one would like to satisfy a constraint of the form

$$X_T^\alpha \in \mathcal{T}, \quad \mathbb{P} - a.s.,$$

where $\mathcal{T} \subset \mathbb{R}^d$ is a given deterministic target. As such the first is an $\mathbb{L}^1$ theory while the stochastic target is an $\mathbb{L}^\infty$ approach.

The stochastic target problem is closely related to stochastic viability theory as developed by Aubin, Cardaliaguet, DaPrato, Frankowska,....
Outline

- Introduction
  - Examples
- Dynamic Programming
- Solutions
  - Black-Scholes
  - Jean - Paul Game
  - Quantile Hedging
  - Martingale Optimal Transport
  - Model Ambiguity
  - Optimal Transport
  - Martingale Optimal Transport
Given are

- **Controlled state process** \( (X_t^{\alpha,x})_{t \geq 0} \in \mathbb{R}^d \) with initial data \( X_0^{\alpha,x} = x \) and an admissible control process \( \alpha \in \mathcal{A} \) defined on a filtered probability space \((\Omega, \{\mathcal{F}_t\})\);
- A set of **probability measures** (or models) \( \mathcal{P} \);
- Desired **success probability** \( z \in [0, 1] \);
- **Deterministic target set** \( \mathcal{T} \subset \mathbb{R}^d \).

Goal is to characterise the reachability set

\[
V(t) := \{ x : \exists \alpha \in \mathcal{A} \text{ so that } \mathbb{P} (X_t^{\alpha,x} \in \mathcal{T}) \geq z, \ \forall \mathbb{P} \in \mathcal{P} \}.
\]
Given are

- **Controlled state process** $(X_t^\alpha,x)_{t \geq 0} \in \mathbb{R}^d$ with initial data $X_0^\alpha,x = x$ and an admissible control process $\alpha \in \mathcal{A}$ defined on a filtered probability space $(\Omega, \{\mathcal{F}_t\})$;
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I concentrate on the case $z = 1$ and $\mathcal{P} = \{\mathbb{P}\}$ at the beginning.
Consider a financial market consisting of a liquidly traded asset with a random price process \( S_u \) and a constant interest rate of \( r \). Assume that \( S \) is a geometric Brownian motion, i.e.,

\[
dS_t = S_t \left[ \mu \, dt + \sigma \, dW_t \right]; \quad t > 0,
\]

where \( \mu > r \), volatility \( \sigma > 0 \) and \( W \) is a Brownian motion.

Further given is a general European option that will pay \( g(S_t) \) at time \( t \) with a known deterministic function \( g \).

Given \( S_0 = s \), consider the function with

\[
\bar{S}_t := e^{rt} S_t,
\]

\[
v(t; s) := \inf \left\{ y : \int_0^t u \, d\bar{S}_u \geq s \right\}.
\]
Consider a financial market consisting of a liquidly traded asset with a random price process $S_u$ and a constant interest rate of $r$. Assume that $S$ is a geometric Brownian motion, i.e.,

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Further given is a general European option that will pay $g(S_t)$ at time $t$ with a known deterministic function $g$.

Given $S_0 = s$, consider the function with $\hat{S}_t := e^{-rt}S_t$,

$$v(t, s) := \inf \left\{ y : \exists \theta \text{ such that } y + \int_0^t \theta_u d\hat{S}_u \geq e^{-rt}g(S_t) \ a.s. \right\}$$
In the notation of the general problem, control is the portfolio process $\theta$ and the state process is

$$
X^{\theta, (y,s)}_t = (Y^\theta_t, S^s_t) = \left( e^{rt} \left[ y + \int_0^t \theta_u d\hat{S}^s_u \right], S^s_t \right),
$$

and the target set is the epigraph of $g$,

$$
\mathcal{T} = \{ (y, s) \in \mathbb{R}^2_+ : y \geq g(s) \}.
$$

Then, the reachability set $V(t)$ is also an epigraph and

$$
v(t, s) = \inf \{ y : (y, s) \in V(t) \}.
$$

The function $v(t, s)$ is the smallest all prices $y$ that allows the seller to hedge the claim $g(S_t)$ with probability one.
Instead of explaining the continuous time stochastic problem, I describe a simple deterministic game. This has the advantage of avoiding technical constructions. It has an immediate stochastic interpretation as well. In continuous time studied by myself and Touzi in (2002). Different derivation in one co-dimension was given by Buckdahn, Cardaliaguet, Quimcampoix (2002) using viability theory. Discrete game was introduced by Kohn & Serfaty in 2006. They studied the small step size limit together with Barles & Da Lio using viscosity solutions.
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In continuous time studied by myself and Touzi in (2002).

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We are given a region \( \mathcal{O} \) on the plane and an initial point \( x_0 \in \mathcal{O} \).

Jane wants to leave the region as quickly as possible and can move one unit step in any direction:

\[
x_1 = x_0 + \nu_1,
\]

where \( \nu_1 \in S^1 \) is chosen by Jane.

Paul can only reverse Jane’s direction and wants to keep her in the region. So he can choose between the two points,

\[
x_1^\pm = x_0 \pm \nu_1.
\]
Jane wants to minimize the exit time.

Paul can be considered a random walk and Jane would like to exit with probability one.
Jane may choose the direction heading directly towards the boundary.
But if she does that, Paul may reserve her and she will end up further to the boundary.
Optimal direction of Jane is to leave Paul irrelevant. This forces Jane to choose the tangential direction rather than the normal. Hence curvature is essential.
Again consider the Black-Scholes structure, i.e.,
\[ dS_t = S_t [\text{dt} + dW_t] \]
and an option pay-off \( g(S_t) \). Instead of hedging with probability one, given \( z \in [0;1] \), one wants to hedge with at least probability \( z \). Using the notation, \( S_0 = s \) and \( \hat{S}_t := e^{rt} S_t \),
\[ v(t; s; z) := \inf \left\{ y : \mathbb{P}(y + \int_0^t u d\hat{S}_u e^{rt} g(S_t) \geq z) \right\} \]
Again consider the Black-Scholes structure, i.e.,

\[ dS_t = S_t[\mu dt + \sigma dW_t] \]

and an option pay-off \( g(S_t) \). Instead of hedging with probability one, given \( z \in [0, 1] \), one wants to hedge with at least probability \( z \). Using the notation, \( S_0 = s \) and \( \hat{S}_t := e^{-rt}S_t \),

\[
v(t, s, z) := \inf \left\{ y : \exists \ \theta \ \text{s.t.} \ \mathbb{P} \left( y + \int_0^t \theta_u d\hat{S}_u \geq e^{-rt}g(S_t) \right) \geq z \right\}.
\]
Here we take one probability measure and $z = 1$, i.e,

$$V(t) := \{ x : \exists \alpha \in \mathcal{A} \text{ so that } X_{t,x}^{\alpha,x} \in \mathcal{T}, \ P - a.s. \}.$$ 

Then, for any $t > 0$ and any stopping time $\tau \leq t$,

$$V(t) = \{ x : \exists \alpha \in \mathcal{A} \text{ so that } X_{\tau,x}^{\alpha,x} \in V(t - \tau), \ P - a.s. \}.$$ 

When $V$ is an epigraph, $v(t, s) = \inf\{ y : (y, s) \in V(t) \}$ satisfies,

$$v(t, s) = \inf\{ y : \exists \alpha \in \mathcal{A} \text{ s.t. } Y_{\tau}^{\alpha,(y,s)} \geq v(t - \tau, S_{\tau}), \ P - a.s. \}.$$ 

In the next slides, I demonstrate this in the simpler discrete example of Jane-Paul game.
Let $u(x)$ be the number steps to leave the region. Then,

$$x \in V(t) \iff u(x) = t.$$ 

$u$ at the point below is 3.
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$u$ at the point below is 3.
There is a simple geometric dynamic programming
The set \( \{ u(x) = 1 \} \) is between the black and the red curves.
The set \( \{u(x) = 2\} \) is between the red and the green curves.
The set \( \{u(x) = 3\} \) is between the purple and the green curves.
Our point moves from \( \{u(x) = 3\} \) into \( \{u(x) = 2\} \) into \( \{u(x) = 1\} \) and then out. This is dynamic programming.
Outline

Introduction

Examples

Dynamic Programming

Solutions

Black-Scholes

Jean - Paul Game

Quantile Hedging

Martingale Optimal Transport

Model Ambiguity

Optimal Transport

Martingale Optimal Transport
Once a dynamic programming principle is established, one uses it to obtain the dynamic programming equation.

In the case of the original target problem, the solution is the reachability set \( V \) and the equation is a geometric equation. In the case of the Jean - Paul game, asymptotically this equation is the mean curvature flow.

In the case of an epigraph, one obtains a parabolic equation.
Introduction

Examples

Dynamic Programming

Solutions

Black-Scholes

Jean - Paul Game

Quantile Hedging

Martingale Optimal Transport

Model Ambiguity

Optimal Transport

Martingale Optimal Transport
Set interest rate $r = 0$. Recall that $dS_u = S_u[\mu du + \sigma dW_u]$,

$$v(s, t) := \inf \left\{ x \mid \exists \theta \text{ admissible } x + \int_0^t \theta_u dS_u \geq g(S_t) \text{ a.s.} \right\}.$$

Dynamic programming states that for any stopping time $\tau$,

$$v(s, t) := \inf \left\{ x \mid \exists \theta \text{ s.t. } x + \int_0^\tau \theta_u dS_u \geq v(t - \tau, S_\tau) \text{ a.s.} \right\}.$$
Set interest rate $r = 0$. Recall that $dS_u = S_u[\mu du + \sigma dW_u]$, 

$$v(s, t) := \inf \left\{ x \mid \exists \theta \text{ admissible } x + \int_0^t \theta_u dS_u \geq g(S_t) \ a.s. \right\}.$$ 

Dynamic programming states that for any stopping time $\tau$, 

$$v(s, t) := \inf \left\{ x \mid \exists \theta \text{ s.t. } x + \int_0^\tau \theta_u dS_u \geq v(t - \tau, S_\tau) \ a.s. \right\}.$$ 

Assume a minimiser $\theta^*$ exists. Then, by Ito calculus, 

$$v(t, s) + \int_0^\tau \theta^*_u dS_u \geq v(t - \tau, S_\tau)$$ 

$$= v(t, s) + \int_0^\tau [v_s(t - u, S_u) dS_u - \mathcal{L}v(t - u, S_u) du],$$ 

$$\mathcal{L}v(t, s) = v_t(t, s) - \frac{s^2 \sigma^2}{2} v_{ss}(t, s).$$
\begin{align*}
v(t, s) + \int_0^T \theta_u^* dS_u & \geq v(t, s) + \int_0^T [v_s(t - u, S_u) dS_u - \mathcal{L}v(t - u, S_u) du], \\
\text{Hence } dS \text{ terms must be equal and we conclude that } \\
\theta_u^* = v_s(t - u, S_u). \\
\text{Moreover } du \text{ term, } \mathcal{L}v, \text{ on the right hand side is zero. Hence, } v \\
is the unique solution of the famous Black-Scholes equation, \\
\begin{align*}
rv + v_t - rsv_s - \frac{s^2 \sigma^2}{2} v_{ss} &= 0, \quad t, s > 0, \\
\text{with initial data } v(0, s) &= g(s).
\end{align*}
\end{align*}
Since $v$ solves

$$rv + v_t - rs v_s - \frac{s^2 \sigma^2}{2} v_{ss} = 0, \quad t, s > 0,$$

with initial data $v(0, s) = g(s)$, Feynman - Kac formula implies that

$$v(t, s) = e^{-rt} \mathbb{E}_{Q^*}[g(S_t^s)],$$

where $Q^*$ is the (unique) probability measure under which $\hat{S}_t = e^{-rt}S_t$ is martingale.
Introduction

Examples

Dynamic Programming

Solutions

Black-Scholes

Jean - Paul Game

Quantile Hedging

Martingale Optimal Transport

Model Ambiguity

Optimal Transport

Martingale Optimal Transport
Recall that $u(x)$ is the minimal time to exit. Then, $u(x) = 0$ outside the target $\mathcal{T}$ and by dynamic programming,

$$u(x) = \inf_{\nu \in S^1} \max \{u(x + \nu), u(x - \nu)\} + 1, \quad x \in \mathcal{T}.$$

This is the dynamic programming equation in this context.

To obtain something more tractable, we do asymptotics by making the step size to be $\epsilon$ rather than one. But then the number of steps required to leave would be large and we define the scaled minimal time function $u^\epsilon(x)$ to be $\epsilon^2$ times the number of required steps.
Using the equation for $u^\epsilon$ and Taylor expansion,

$$u^\epsilon(x) = \inf_{\nu \in S^1} \max \{ u^\epsilon(x + \epsilon \nu), u^\epsilon(x - \epsilon \nu) \} + \epsilon^2$$

$$\approx u^\epsilon(x) + \inf_{\nu \in S^1} \max \left\{ \pm \epsilon \nu \cdot \nabla u^\epsilon(x) + \frac{\epsilon^2}{2} D^2 u^\epsilon(x) \nu \cdot \nu \right\} + \epsilon^2.$$  

Then, optimal $\nu^*$ satisfies

$$\nu^* \cdot \nabla u^\epsilon(x) = 0, \quad \Rightarrow \quad \nu^* = \frac{(\nabla u^\epsilon(x))^\perp}{|\nabla u^\epsilon(x)|}.$$  

Moreover, formally the limit function $u$ should satisfy (using the $\epsilon^2$ terms),

$$\frac{1}{2} \left[ \Delta u - \frac{D^2 u(x) \nabla u(x) \cdot \nabla u(x)}{|\nabla u(x)|^2} \right] + 1 = 0, \quad x \in \mathcal{T}.$$  

This is the mean curvature flow equation.
One can define the game directly in continuous time as well.

\[
V(t) = \{ x \in \mathbb{R}^2 : \exists \{ \nu_u \}_{u \in [0,t]} \in S^1 \text{ s.t. } \chi^{x,\nu}_t \in \mathcal{T} \},
\]

where

\[
\chi^{x,\nu}_t := x + \int_0^t \nu_u (\nu_u \cdot dW_u)
\]

and \( W \) is a standard two dimensional Brownian motion.

Multi dimensional versions with higher co-dimension is also available.
Introduction

Examples

Dynamic Programming

Solutions

Black-Scholes

Jean-Paul Game

Quantile Hedging

Martingale Optimal Transport

Model Ambiguity

Optimal Transport

Martingale Optimal Transport
Recall $dS_t = S_t[\mu dt + \sigma dW_t]$ and set $r = 0$. Given desired success probability $z \in [0, 1]$ and $S_0 = s$,

$$v(t, s, z) := \inf \left\{ y : \exists \theta \text{ s.t. } \mathbb{P} \left( y + \int_0^t \theta_u dS_u \geq g(S_t) \right) \geq z \right\}.$$ 

With fixed $z$, the above does not satisfy the dynamic programming unless $z = 1$.

However, Bouchard, Elie, Touzi transformed the above problem to a problem with $z = 1$ but with one more state variable accounting for the success probability.
A process $\theta$ is admissible if $Y^\theta_t(y,s) := y + \int_0^t \theta_u dS_u^s \geq 0$. Set

$$Z_u := \mathbb{P} \left( Y^\theta_t(y,s) \geq g(S_t) \mid \mathcal{F}_u \right), \quad u \in [0, t].$$

Then, since $\mathcal{F}_0$ is trivial, $Z_0 = z$ and

$$Z_t = \chi_{\{Y^\theta_t(y,s) \geq g(S_t^s)\}}.$$

Hence, $Z$ is a martingale. By the martingale representation Theorem, there exist a process $\alpha$ so that

$$Z_t = z + \int_0^t \alpha_u dW_u =: Z^{\alpha,z}_t.$$

Then, the original quantile hedging problem is equivalent to

$$v(t, s, z) = \inf \left\{ y : \exists (\theta, \alpha) \text{ s.t. } Z^{\alpha,z}_t \geq \chi_{\{Y^\theta_t(y,s) \geq g(S_t^s)\}}, \mathbb{P} - \text{a.s.} \right\}.$$
Consider quantile hedging in a Binomial model with equal up and down probabilities of 1/2. This means

\[ S_{n+1} = S_n \ [1 + \xi_{n+1}], \quad \xi_{n+1} = \pm 1 \text{ with equal probability.} \]
When $z \in (0.75, 1]$ we cannot miss any nodes, $z \in (0.5, 0.75]$ we are allowed to miss one of the nodes, $z \in (0.25, 0.5]$ we are allowed to miss two nodes, ...
$S_{n+1} = S_n[1 + \xi_{n+1}]$ and the probability process is given by

\[ Z_{n+1} = Z_n + \alpha_n\xi_{n+1}. \]
Consider the case $z = 0.5$. So we can choose 2 nodes to miss.
Suppose we decide to miss the middle 2. Then, the probabilities \( Z \) evolve as below.

\[
\begin{align*}
Z_0 &= 0.5 \\
Z_1 &= 0.5 \\
Z_1 &= 0.5 \\
Z_2 &= 1 \\
Z_2 &= 0 \\
Z_2 &= 0 \\
Z_2 &= 1
\end{align*}
\]
If we decide to miss the lower two, then the probabilities $Z$ evolve as below.

$Z_0 = 0.5$
$Z_1 = 1$
$Z_2 = 1$

$Z_0 = 0.5$
$Z_1 = 0$
$Z_2 = 0$
\[ S_{n+1} = S_n [1 + \xi_{n+1}], \quad Z_{n+1} = Z_n + \alpha_n \xi_{n+1}, \]

In the first one, \( \alpha \)'s are given as below.

\[ \begin{align*}
\alpha_0 &= 0 \\
Z_0 &= 0.5 \\
\alpha_1 &= 1/2 \\
Z_1 &= 0.5 \\
\alpha_1 &= -1/2 \\
Z_1 &= 0.5 \\
\end{align*} \]

\[ \begin{align*}
\cdot Z_2 &= 1 \\
\cdot Z_2 &= 0 \\
\cdot Z_2 &= 0 \\
\cdot Z_2 &= 1 \\
\end{align*} \]
\[ S_{n+1} = S_n[1 + \xi_{n+1}], \quad Z_{n+1} = Z_n + \alpha_n\xi_{n+1}, \]

In the second one, \(\alpha\)'s are given as below.

\[
\begin{align*}
\alpha_0 &= 1/2 \\
Z_0 &= 0.5 \\
\alpha_1 &= 0 \\
Z_1 &= 1 \\
\alpha_1 &= 0 \\
Z_1 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
\bullet Z_2 &= 1 \\
\bullet Z_2 &= 0
\end{align*}
\]
Introduction
Examples
Dynamic Programming
Solutions
Black-Scholes
Jean - Paul Game
Quantile Hedging
Martingale Optimal Transport
Model Ambiguity
Optimal Transport
Martingale Optimal Transport
Consider again the Black-Scholes model, with \( r = 0 \) and 
\[
dS_t^\sigma = S_t^\sigma [\mu dt + \sigma_t dW_t].
\]
Here we assume that \( \sigma_t \) is not constant and can be any adapted process in a given interval \([\sigma, \overline{\sigma}]\). Let \( \mathcal{A} \) be the set of all such processes. For a given \( \mathcal{F}_t \) measurable, bounded \( \xi \), set

\[
v(\xi) := \inf \left\{ y : \exists \theta \text{ s.t. } y + \int_0^t \theta_u dS_u^\sigma \geq \xi \text{ a.s., } \forall \sigma. \in \mathcal{A} \right\}.
\]

One could show (rather technical) that

\[
v(\xi) := \sup_{\sigma. \in \mathcal{A}} \inf \left\{ y : \exists \theta \text{ s.t. } y + \int_0^t \theta_u dS_u^\sigma \geq \xi \text{ a.s. } \right\}.
\]
Recall the Black-Scholes with $r = 0$,

$$\nu(s, t) := \inf \left\{ x \mid \exists \theta \text{ admissible} \ x + \int_0^t \theta_u dS_u \geq \xi \ \text{a.s.} \right\},$$

where $\xi$ is a given bounded $\mathcal{F}_t$ measurable random variable. Then,

$$\nu(t, s) = \mathbb{E}_{\mathbb{Q}^*} [\xi],$$

where $\mathbb{Q}^*$ is the (unique) probability measure under which $S_t$ is martingale.
\[ dS_t^\sigma = S_t^\sigma [\mu dt + \sigma_t dW_t], \quad \sigma_t \in [\underline{\sigma}, \overline{\sigma}], \quad \mathcal{A} \text{ be the set of all such processes and } \xi \text{ is } \mathcal{F}_t \text{ measurable, bounded. Using the result from Black-Scholes,} \]

\[
v(\xi) := \inf \left\{ y : \exists \theta \text{ s.t. } y + \int_0^t \theta_u dS_u^\sigma \geq \xi \text{ a.s., } \forall \sigma \in \mathcal{A} \right\}
\]

\[
= \sup_{\sigma \in \mathcal{A}} \inf \left\{ y : \exists \theta \text{ s.t. } y + \int_0^t \theta_u dS_u^\sigma \geq \xi \text{ a.s., } \right\}
\]

\[
= \sup_{\mathcal{Q} \in \mathcal{M}} \mathbb{E}_{\mathcal{Q}}[\xi],
\]

where \( \mathcal{Q} \in \mathcal{M} \) if and only if \( \mathcal{Q} \) is the “distribution” of \( \bar{S}^\sigma \) for some \( \sigma \in \mathcal{A} \) and \( d\bar{S}_u^\sigma = \bar{S}_u^\sigma \sigma_u dW_u. \)
Introduction

Examples

Dynamic Programming

Solutions

Black-Scholes

Jean-Paul Game

Quantile Hedging

Martingale Optimal Transport

Model Ambiguity

Optimal Transport

Martingale Optimal Transport
In the previous problem, if we take the set of probability measures $\mathcal{Q}$ to be the whole set this corresponds to complete model independence. However, then for many functions $\xi$, we might have

$$\sup_{\mathcal{Q} \in \mathcal{Q}} \mathbb{E}[\xi] = \|\xi\|_\infty,$$

and the problem trivialises. But a version of this problem is very close to the celebrated Monge-Kantarovich optimal transport problem.
The Monge-Kantorovich problem is in discrete time with two time steps and its dual, derived by Kantarovich is given by,

\[ \nu(\xi) := \sup \left\{ \int_{\mathbb{R}^d} hd\mu + \int_{\mathbb{R}^d} gd\nu : h(x) + g(y) \geq \xi(x, y), \forall (x, y) \in \mathbb{R}^{2d} \right\}, \]

where \( \mu, \nu \) are given probability measures on \( \mathbb{R}^d \) and \( \xi \) is bounded function of \( \mathbb{R}^{2d} \).
\[ v(\xi) := \sup \left\{ \int_{\mathbb{R}^d} h \, d\mu + \int_{\mathbb{R}^d} g \, d\nu : h(x) + g(y) \geq \xi(x, y), \forall (x, y) \in \mathbb{R}^{2d} \right\}, \]

The primal is the celebrated optimal transport problem,

\[ v(\xi) = \sup_{Q \in \mathcal{Q}(\mu, \nu)} \mathbb{E}_Q[\xi], \]

where \( Q \in \mathcal{Q}(\mu, \nu) \) if it is a probability measure whose marginals are \( \mu \) and \( \nu \), respectively.
Introduction

Examples

Dynamic Programming

Solutions

Black-Scholes

Jean - Paul Game

Quantile Hedging

Martingale Optimal Transport

Model Ambiguity

Optimal Transport

Martingale Optimal Transport
We consider a direct generalisation of the Monge-Kantorovich problem motivated by a financial problem, the so-called forward-start options. The model independent version of this problem is

\[ v(\xi) := \sup \left\{ \int_{\mathbb{R}^d} h \, d\mu + \int_{\mathbb{R}^d} g \, d\nu : \exists \gamma : \mathbb{R}^d \to \mathbb{R}^d \right. \]

\[ h(x) + g(y) + \gamma(x) \cdot (y - x) \geq \xi(x, y), \quad \forall (x, y) \in \mathbb{R}^{2d} \]
The dual is

\[ v(\xi) := \sup \left\{ \int_{\mathbb{R}^d} h d\mu + \int_{\mathbb{R}^d} g d\nu : \exists \gamma : \mathbb{R}^d \to \mathbb{R}^d \right. \]

\[ h(x) + g(y) + \gamma(x) \cdot (y - x) \geq \xi(x, y), \quad \forall (x, y) \in \mathbb{R}^{2d} \}

\[ = \sup_{\mathcal{Q} \in \mathcal{M}(\mu, \nu)} \mathbb{E}_\mathcal{Q}[\xi], \]

where \( \mathcal{Q} \in \mathcal{M}(\mu, \nu) \) if it is in \( \mathcal{Q}(\mu, \nu) \) (i.e., satisfies the marginals) and also it is a martingale measure, i.e.,

\[ \mathbb{E}_\mathcal{Q}[\gamma(x) \cdot (y - x)] = 0, \]

for any bounded measurable \( \gamma \).
If we see $y$ as the value of a process at time 2, $S_2$, and $x$ as $S_1$, then above simply states that

$$\mathbb{E}_Q[\gamma(x) \cdot (y - x)] = 0, \quad \forall \gamma, \quad \iff \quad \mathbb{E}_Q[S_2 | S_1] = S_1.$$ 

Hence, $S$ is a martingale under the measure $Q$. This new constraint is due to the term $\gamma(x) \cdot (y - x)$ in the definition of the martingale optimal transport problem.
In uncertain volatility,

\[ \nu(\xi) = \sup_{Q \in \mathcal{M}} \mathbb{E}_{Q}[\xi], \]

where \( \mathcal{M} \) is a subset of martingale measures.

In optimal transport,

\[ \nu(\xi) = \sup_{Q \in \mathcal{Q}(\mu, \nu)} \mathbb{E}_{Q}[\xi], \]

where \( \mathcal{Q}(\mu, \nu) \) satisfies marginal constraints.

In martingale optimal transport,

\[ \nu(\xi) = \sup_{Q \in \mathcal{M}(\mu, \nu)} \mathbb{E}_{Q}[\xi], \]

where \( \mathcal{M}(\mu, \nu) \) satisfies marginal and martingale constraints.
Let $\Omega$ be the Skorokhod space of all $\mathbb{R}^d_+$-valued càdlàg processes on $[0, t]$, i.e., process that are continuous from right hand have left limits with the standard Skorokhod topology.

Further let $S$ be the canonical map, i.e. $S_u(\omega) = \omega_u$ for all $u \in [0, t]$ and $\mathcal{F}_u$ be the filtration generated by $S$. Given a probability measure $\nu$ on $\mathbb{R}^d_+$, set

$$
\nu(\xi) := \sup \left\{ \int_{\mathbb{R}^d} g \, d\nu : \exists (\gamma_u)_{u \in [0, t]} \right\}
$$

$$
g(S_t(\omega)) + \int_0^t \gamma_u(\omega) \cdot dS_u(\omega) \geq \xi(\omega), \ \forall \omega \in \Omega \right\}.
$$

Assume that $\xi$ is bounded and uniformly continuous with respect to the Skorokhod metric. Then,

$$\nu(\xi) = \sup_{Q \in \mathcal{M}(\mu)} \mathbb{E}_Q[\xi].$$

The set $\mathcal{M}(\mu)$ is the set of all measures $Q$ under which the canonical map $S$ is a martingale and the marginal of $Q$ at time $t$ is $\nu$. Note that it is not compact.

Many marginal version is also proved.
The stochastic target type problems can be solved by geometric dynamic principle.

The quantile type problems can be transferred into a standard target problems.

One could also consider cases in which many possibly orthogonal measures are given representing the model uncertainty.

THANK YOU FOR YOUR ATTENTION.