Going beyond mean-field approximations

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Model

Interacting Markov chains, or diffusions, on a sparse graph $G = (V, E)$:

$$
\begin{aligned}
X_v(k+1) &= F(X_v(k), \mu_v(k), \xi_v(k+1)), && k \in \mathbb{N}_0, \\
dX_v(t) &= b(X_v(t), \mu_v(t))\, dt + \sigma(X_v(t), \mu_v(t))\, dW_v(t), && t \geq 0,
\end{aligned}
$$

where $\mu_v(k)$ is the neighborhood empirical measure given by

$$
\mu_v(k) = \frac{1}{|N_v(G)|} \sum_{u \in N_v(G)} \delta_{X_u(k)},
$$

and $N_v(G) := \{u \in V : (u, v) \in E\}$ denotes the neighborhood of v.

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Results to discuss:

(a) 2nd order Markov random fields $\longrightarrow$ local-field equations

(b) Correlation decay $\longrightarrow$ convergence of empirical measures

(c) Dense graphs with heterogeneous limits $\longrightarrow$ graphon particle systems

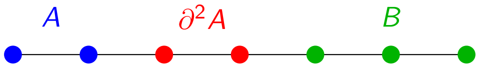
2nd-order Markov random fields

Definition: A family of random variables $(Y_v)_{v \in G}$ is a 2nd-order Markov random field if

$$(Y_v)_{v \in A} \perp (Y_v)_{v \in B} \;\big|\; (Y_v)_{v \in \partial^2 A},$$

for all sets $A, B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

Example:



Notation: For a set A of vertices in a graph $G = (V, E)$, define

$$\text{Boundary: } \partial A = \{u \in V : d(u, A) = 1\},$$

$$\text{Double-boundary: } \partial^2 A = \{u \in V : d(u, A) = 1, 2\} = \partial A \cup \partial(A \cup \partial A).$$

2nd-order Markov random fields

Theorem (Lacker-Ramanan-W. '25)

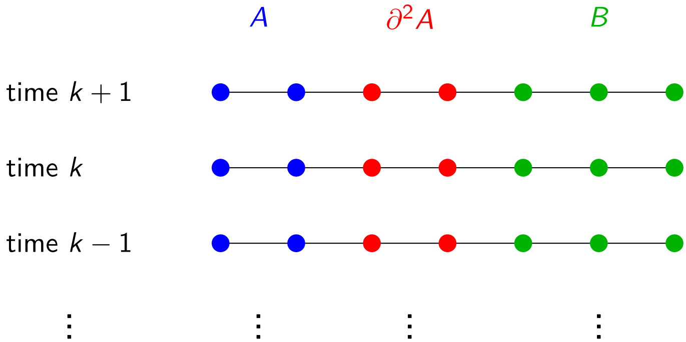
Assume the initial states $(X_v(0))_{v \in V}$ form a second-order MRF. For each time t, the particle trajectories $(X_v[k])_{v \in V}$ form a second-order MRF. Here $x[k] = (x(0), x(1), \ldots, x(k))$.

This is sharp: in general, for $k \geq 1$,

- $(X_v[k])_{v \in V}$ is not a first-order MRF.
- $(X_v(k))_{v \in V}$ is not a MRF of any order.

2nd-order Markov random fields

Why? We need to condition on the paths of two nodes in the middle to cut off the influence between A and B.



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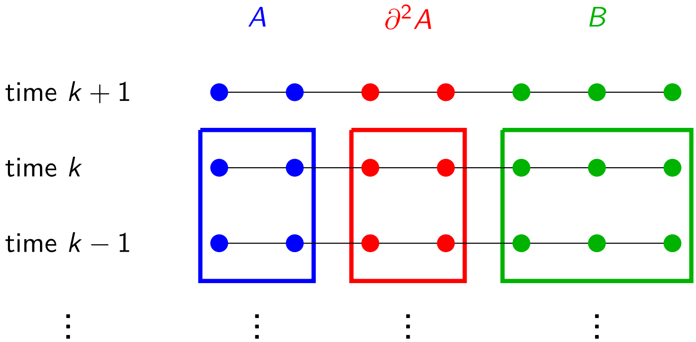

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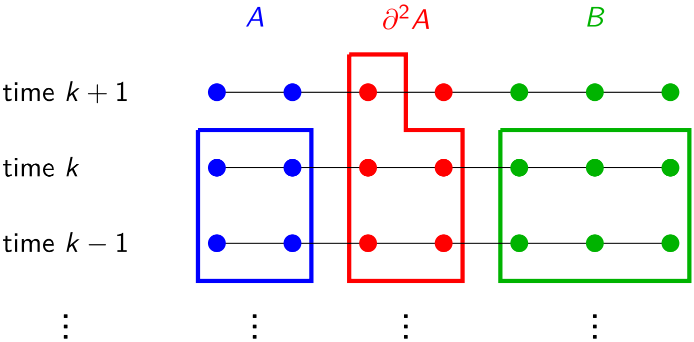

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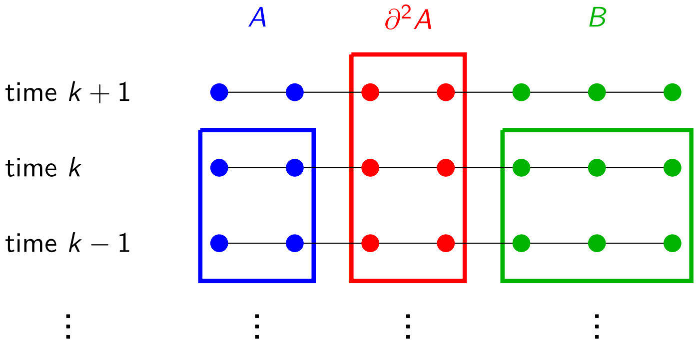

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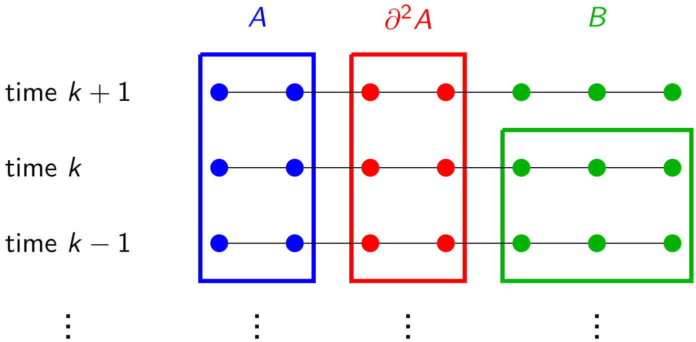

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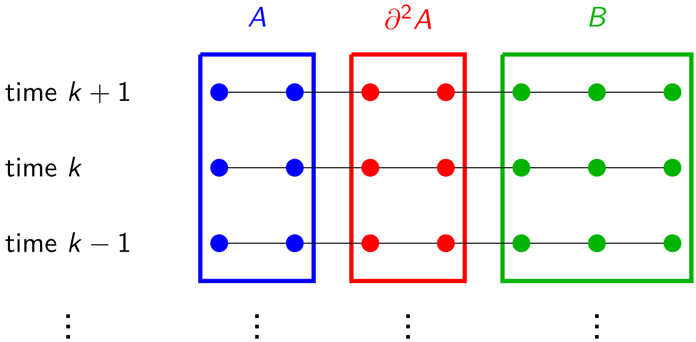


Correlation decay

Theorem (Lacker-Ramanan-W. '23)

Under suitable conditions, there exists $c\colon \mathbb{N}_0 \to \mathbb{R}_+$ *with* $\lim_{n\to\infty} c(n) = 0$ *such that*

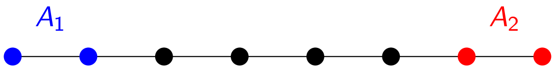
$$|\mathrm{Cov}(f_1(X_{A_1}[T]), f_2(X_{A_2}[T]))| \leq (|A_1| + |A_2|)\, c(d_G(A_1, A_2))$$

for all functions f_1, f_2 *that are* 1*-Lipschitz and bounded by* 1*.*

Essential for proofs of convergence of empirical measures.

Correlation decay

Argue via coupling: $\mathrm{Cov}(X_{A_1}, X_{A_2}) = \mathbb{E}[X_{A_1} X_{A_2}] - \mathbb{E}[X_{A_1}]\mathbb{E}[X_{A_2}]$



Will construct Y and Z such that

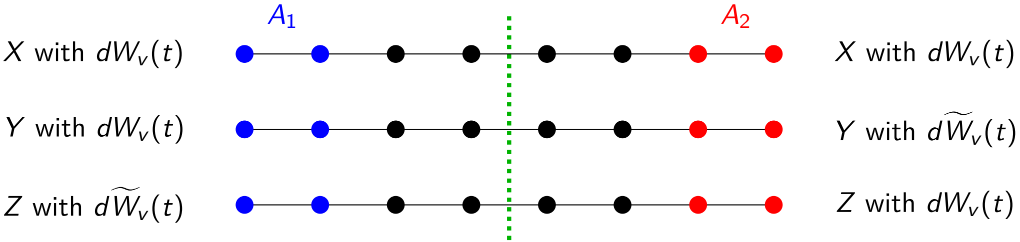
- Y and Z are independent;
- $Y \stackrel{d}{=} X \stackrel{d}{=} Z$;
- $\mathbb{E}|Y_{A_1} - X_{A_1}|$ and $\mathbb{E}|Z_{A_2} - X_{A_2}|$ decay as $d_G(A_1, A_2)$ grows.

Then

$$\mathbb{E}[X_{A_1} X_{A_2}] - \mathbb{E}[X_{A_1}]\mathbb{E}[X_{A_2}] = \mathbb{E}[X_{A_1} X_{A_2}] - \mathbb{E}[Y_{A_1} Z_{A_2}] \leq C\mathbb{E}|Y_{A_1} - X_{A_1}| + C\mathbb{E}|Z_{A_2} - X_{A_2}|.$$

Correlation decay

Replace some of the driving Brownian motion W by independent copies $\widetilde{W}$:



By construction: Y and Z are independent; $Y \overset{d}{=} X \overset{d}{=} Z$. Similar to Picard iteration estimates:

$$\max_{v:\, d(v,A_1)\le k} \mathbb{E}\left[\|X_v - Y_v\|^2_{*,t}\right] \le C \int_0^t \max_{v:\, d(v,A_1)\le k+1} \mathbb{E}\left[\|X_v - Y_v\|^2_{*,s}\right] ds$$

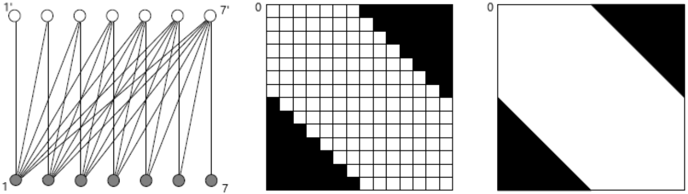
$$\implies \max_{v\in A_1} \mathbb{E}\left[\|X_v - Y_v\|^2_{*,t}\right] \le (Ct)^L / L!, \qquad L = \lceil d_G(A_1, A_2)/2 \rceil.$$

Dense graphs $\to$ graphon

Particles with heterogeneous interaction $\xi_{ij}^n \in [0,1]$ corresponding to $G_n \colon [0,1]^2 \to [0,1]$

$$X_i^n(t) = x_0 + \int_0^t \frac{1}{n} \sum_{j=1}^n \xi_{ij}^n b(X_i^n(s), X_j^n(s))\, ds + W_i(t).$$

Example: Half-graph and its limit.



Limit of half-graphs in the cut norm $\|\cdot\|_\square$.

(Lovász '12)

The cut metric is weak: $\|G\|_\square \le \|G\|_{L_1} \le \|G\|_{L_2}$, but convenient for convergence of graphs.

E.g., $G_n = \text{Erdős–Rényi}(n, \frac{1}{2}) \to G \equiv \frac{1}{2}$ in $\|\cdot\|_\square$ but not in L_p, since $|G_n(u,v) - G(u,v)| \equiv \frac{1}{2}$.

Dense graphs → graphon

We expect and can show

$$X_i^n(t) = x_0 + \int_0^t \frac{1}{n} \sum_{j=1}^n \xi_{ij}^n b(X_i^n(s), X_j^n(s))\, ds + B_{\frac{i}{n}}(t),$$

$$\implies \qquad X_u(t) = x_0 + \int_0^t \int_{[0,1]} \int_{\mathbb{R}^d} b(X_u(s), x) G(u, v)\, \mu_{v,s}(dx)\, dv\, ds + B_u(t),$$

$$\mu_{v,t} = \mathcal{L}(X_v(t)), \quad v \in [0,1].$$

Theorem (Bayraktar-Chakraborty-W. '23)

Under conditions, $\mu^n := \frac{1}{n}\sum_{i=1}^n \delta_{X_i^n} \to \bar{\mu} := \int_0^1 \mathcal{L}(X_u)\, du$ *in* $\mathcal{P}(\mathcal{C}_d)$ *in probability as* $n \to \infty$.

No continuity of the limiting graph G is needed.

Dense graphs $\to$ graphon

Graphon particle systems are also analyzed for ...

- jump processes
  - SIR model with heterogeneous connection
  - JSQ-d with dispatchers and servers connected via heterogeneous bipartite graphs
- games
- control problems
- etc.

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