Dynamic Contracts:
A Continuous-Time Approach

Yuliy Sannikov
Stanford University
Plan

• Background: principal-agent models in economics
  – Examples: what questions are economists interested in?

• Continuous-time approach to dynamic agency
  – Basic model – double optimization
  – Agent’s incentives
  – Principal’s control problem
  – Characteristics of solution
  – Going further: persistent private information, hidden savings, etc

• Open questions
Principal-agent models in economics

• Corporate finance
  – Capital structure and financing constraints
  – Executive compensation
• Macroeconomics / public finance
  – Taxation / insurance / incentives
• Personnel economics
  – Labor contracts
• Industrial organizations
Basic model

• Phelan and Townsend (1991): time $t = 0, 1, 2, \ldots$
• Probability of output given effort: $P(x | a)$
• In period $t$, principal observes $x^t = \{x_0, x_1, \ldots x_t\}$, but not \{a_0, a_1, \ldots a_t\}, pays the agent $c_t(x^t)$
• Problem: design contract \{c_t(x^t)\} to maximize

$$E\left[ \sum_{t=0}^{\infty} \delta^t (x_t - c_t) \right]$$

s.t. $$E\left[ \sum_{t=0}^{\infty} \delta^t u(a_t, c_t) \right] = w$$

\{a_t\} maximizes $$E\left[ \sum_{t=0}^{\infty} \delta^t u(a_t, c_t) \right]$$ given $c_t(x^t)$
Optimal taxation: the Mirrlees model

- Taxpayers’ abilities \( \sim f(\theta) \), \( \theta \) is wage, choose labor \( l \)
- Gov’t observes only income \( \theta l \), but not \( \theta, l \) separately
- Agents get utility \( u(c(\theta), l), c(\theta) = \theta l - T(\theta l) \)
- Choose tax policy \( T() \) to maximize

Social welfare function 

\[
\int G(u(c(\theta), l(\theta))) f(\theta) d\theta
\]

s.t. (budget) 

\[
\int T(\theta l(\theta)) f(\theta) d\theta = R
\]

\( (IC) \) \( l(\theta) = \arg \max u(\theta l - T(\theta l), l) \)

- Dynamic models (Werning, Farhi, Golosov, Tsyvinski) – stochastic process for ability \( \theta \), investment in human capital, etc.
Dynamic price discrimination

- Battaglini (AER 2005)
- Buyer’s per-period utility: $\theta_t q_t - p_t$
- Seller’s cost: $q_t^2 / 2$
- Buyer’s type $\theta_t$: Markov process (private info)
- In period $t$, seller offers any $q$ at price $p(q)$ (schedule depends on entire past history)
- Objective: maximize profit

$$E\left[ \sum_{t=0}^{\infty} \delta^t (p_t - q_t^2 / 2) \right]$$

s.t. (1) buyer’s expected utility $\geq 0$ and
(2) sequence $q_t$ maximizes buyer’s utility given pricing policy and type sequence $\theta_t$
Basic Theory

  – Patient agent → efficiency is attainable
  – Optimal contract is recursive (agent’s continuation value is the state variable that controls incentives)

• **Why continuous time?** Discrete-time models are messy – full of details that distract from big picture
  – E.g. in Phelan and Townsend (1991), optimal contract involves randomizations – before the agent puts in effort, principal randomizes over which effort he wants to incentivize
  – Agent’s payoff follows a random walk with many step sizes – as many as output levels – distracts from long-term distribution properties
Applications (using continuous-time approach)

- DeMarzo and Sannikov (JF 2006): corporate finance application, unobservable cash flows, capital structure question
- Biais, Mariotti, Plantin and Rochet (ReStud 2007), He (JFE 2007), Biais, Mariotti, Rochet and Villeneuve (EMA 2010), Hoffmann and Pfeil (RFS 2010), Piskorski and Tchistyi (RFS 2010, 2011), DeMarzo, Fishman, He and Wang (JF 2012), DeMarzo and Sannikov (ReStud 2017), He, Wei, Yu and Gao (RFS 2017)

- Optimal taxation / insurance: Farhi and Werning (ReStud 2013), Williams (EMA 2011)

- Repeated Games: Sannikov (EMA 2006)
- Games with reputation: Faingold and Sannikov (EMA 2011)
Basic Model (continuous time)

• Time \( t \in [0, \infty) \)
• Risk-neutral principal and risk-averse agent, common discount rate \( r \)
• Agent puts in effort \( A = \{ A_t \in \mathcal{A}, 0 \leq t < \infty \} \)
• Principal does not see effort, but observes output
  \[
  dX_t = A_t \, dt + \sigma dZ_t
  \]
  where \( Z \) is a Brownian motion
• Cost of effort \( h : \mathcal{A} \to \mathbb{R} \) continuous, increasing, convex
  with \( h(0) = 0 \).
• Utility of consumption \( u : [0, \infty) \to [0, \infty) \) continuous, increasing and concave with
  \( u(0) = 0 \) and
  \( u'(c) \to 0 \) as \( c \to \infty \).

Based on Sannikov (2008) “A Continuous-Time Version of the Principal-Agent Problem,” ReStud
Problem Formulation

Find a contract \( \{C_t, \, 0 \leq t < \infty\} \) and effort recommendation \( A = \{A_t \in A, \, 0 \leq t < \infty\} \) that maximizes the principal’s profit

\[
E^A \left[ r \int_0^\infty e^{-rt} (dX_t - C_t) \, dt \right]
\]

subject to

\[
W_0 = E^A \left[ r \int_0^\infty e^{-rt} (u(C_t) - h(A_t)) \, dt \right]
\]

and

\[
W_0 \geq E^\hat{A} \left[ r \int_0^\infty e^{-rt} (u(C_t) - h(\hat{A}_t)) \, dt \right]
\]
Nature of the problem

• Two embedded dynamic optimization problems
  – Principal is offering a dynamic contract recognizing that the agent will be optimizing dynamically

• To solve:
  – introduce new process – agent’s continuation payoff
    – to characterize the agent’s incentives
  – this reduces the principal’s problem to optimal stochastic control
4 steps to derive the optimal contract

1. Define the agent’s continuation value \((W_t)_{t \geq 0}\) for any \(\{C_t, 0 \leq t < \infty\}\) where \(A = \{A_t \in A, 0 \leq t < \infty\}\)

2. Using the Martingale Representation Theorem, write an equation that describes the evolution of \(W_t\)

3. Necessary and sufficient conditions for the agent’s effort to be optimal (sensitivity of \(W_t\) to \(X_t\))

4. Solve the principal’s problem using the HJB equation
The Agent’s Continuation Value $W_t$

- **Step 1:** Given consumption $\{C_s, 0 \leq s < \infty\}$ and effort $\{A_s, 0 \leq s < \infty\}$, define the agent’s continuation value

$$W_t = E_t \left[ r \int_t^\infty e^{-r(s-t)} (u(C_s) - h(A_s)) ds \right]$$
The Agent’s Continuation Value $W_t$

- **Step 2:**

**Proposition 1.** In any contract with finite payoff to the agent, $W_t$ is the agent’s continuation value if and only if

$$dW_t = r(W_t - u(C_t) + h(A_t))dt + rY_t(dX_t - A_t dt).$$

for some process $\{Y_t\}$ and $E[e^{-rt}W_t] \to 0$.

**Sketch of proof.** Existence of $Y_t$ follows from the representation of

$$r \int_0^t e^{-rs} (u(C_s) - h(A_s)) ds + e^{-rt}W_t = E_t \left[ r \int_0^\infty e^{-rs} (u(C_s) - h(A_s)) ds \right]$$

using Martingale Representation Theorem.
Incentives

The agent maximizes

\[ E[r(u(C_t) - h(\hat{A}_t))dt + dW_t] \]

\[ dW_t = r(W_t - u(C_t) + h(A_t))dt + rY_t(dX_t - A_t dt) \]

Agent will maximize \[ Y_t\hat{A}_t - h(\hat{A}_t) \]

**Step 3:**

**Proposition 2.** A contract is incentive-compatible if and only if

\[ \forall t \geq 0, A_t \text{ maximizes } Y_t a - h(a) \] (IC)

If (IC) holds, we say \( Y_t \) enforces \( A_t \).
Sketch of Proof

Consider

\[ \hat{V}_t = r \int_0^t e^{-rs} (u(C_s) - h(\hat{A}_s)) ds + e^{-rt} W_t \]

If (IC) holds, then for any deviation, \( \hat{V}_t \) is an \( \hat{A} \)-supermartingale. Hence,

\[ W_0 = \hat{V}_0 \geq E^{\hat{A}} \left[ r \int_0^t e^{-rs} (u(C_s) - h(\hat{A}_s)) ds + e^{-rt} W_t \right], \]

and taking \( t \) to infinity, we find that \( A \) is not worse than \( \hat{A} \).

If (IC) fails, we can find a deviation such that \( \hat{V}_t \) is an \( \hat{A} \)-submartingale. Hence,

\[ W_0 = \hat{V}_0 < E^{\hat{A}} \left[ r \int_0^t e^{-rs} (u(C_s) - h(\hat{A}_s)) ds + e^{-rt} W_t \right] \]
The Optimal Control Problem

Propositions 1 and 2 imply:

**Theorem:** There is a one-to-one correspondence between

- Contracts \( \{C_t, t \geq 0\} \) with strategies \( \{A_t, t \geq 0\} \) that satisfy the incentive constraints, with finite value to the agent and

- Controlled processes

\[
dW_t = r(W_t - u(C_t) + h(A_t))dt + rY_t \sigma dZ_t
\]

that satisfy the transversality condition \( \lim_{t \to \infty} E[e^{-rt}W_t] = 0 \), with controls \( \{C_t, A_t, Y_t\} \) such that \( Y_t \) enforces \( A_t \)
The Optimal Control Problem

\[ dW_t = r(W_t - u(C_t) + h(A_t)) dt + rY_t \sigma dZ_t \]

• The principal
  – controls \( W_t \) with \( C_t, A_t \) and \( Y_t \) (which enforces \( A_t \))
  – must honor promises, i.e. \( E[e^{-rt}W_t] \to 0 \)
  – gets expected flow of profit of \( A_t - C_t \)

Denote by \( F(W_0) \) the maximal total profit that the principal can attain in this way
HJB equation

Controlled process:

\[ dW_t = r(W_t - u(C_t) + h(A_t))dt + rY_t\sigma\,dZ_t \]

HJB equation:

\[
rF(W) = \max_{c,a,Y\text{ s.t.}} r(a - c) + \]

\[
r(W - u(c) + h(a))F'(W) + \frac{rY^2\sigma^2}{2}F''(W) \]

Denote by \( y(a) \) minimal (in absolute value) \( Y \) that enforces a
Profit function $F$ vs. first best

Let $F_0(u(c)) = -c$ be “retirement” profit. Solve

$$F''(W) = \min_{a>0,c} \frac{F(W) + c - a - (W - u(c) + h(a))F'(W)}{r\gamma(a)^2\sigma^2/2}$$

with $F(0) = 0$, and largest $F'(0)$ such that $F(W_{gp}) = F_0(W_{gp})$ for some $W_{gp} \geq 0$
The Optimal Contract

$F(W_0)$ is the principal’s profit in the optimal contract for $W_0 \in [0, W^{gp}]$. The agent’s value in the optimal contract follows

$$dW_t = r(W_t - u(c(W_t)) + h(a(W_t))) \ dt + rY(W_t) \left( dX_t - a(W_t) dt \right),$$

until the retirement time $\tau$ when $W_t$ hits 0 or $W^{gp}$. For $t < \tau$, $C_t = c(W_t)$ and $A_t = a(W_t)$ are the maximizers in the ODE for $F(W)$. After time $\tau$, the agent receives constant consumption $C_t = -F(W_\tau)$ and puts effort 0.
An Example

\[ u(c) = \sqrt{c}, \quad h(a) = 0.5a^2 + 0.4a, \quad r = 0.1 \text{ and } \sigma = 1. \]
Properties of solution

• Cont. value $W_t$ fully summarizes past history
  – bad performance history can be undone by good performance (unless the agent is retired at 0)

• It’s optimal to eventually let agent retire (at $0/W_{gp}$)

• Contract is not renegotiation-proof
  – $F(W)$ has an increasing portion: principal punishes the agent at a cost

• Payoff $W^*$ that maximizes $F(W)$ is positive
  – principal needs room to reward and punish the agent
Where can we go from here?

• Easy: change boundary conditions
  • The agent’s outside option
  • The cost of replacing the agent
  • Promotion opportunities

• Harder: need to add more state variables to summarize the agent’s incentives
  – Agent’s effort affects output now and in the future
  – Agent privately observes persistent shocks to output
  – Hidden savings
Example: asset management + hidden savings

- Agent manages capital $k_t$, obtains return per dollar $dR_t = (\alpha + r - a_t) dt + \sigma dZ_t$

- Agent’s utility
  $$W^{a,c} = E^a \left[ \int_0^\infty e^{-rt} \frac{\hat{C}_{t}^{1-\gamma}}{1-\gamma} dt \right]$$

- $a_t \geq 0$ is “stealing”. Hidden savings, $h_t \geq 0$
  $$dh_t = \left( rh_t + C_t - \hat{C}_t + \phi k_t a_t \right) dt, \quad \phi \leq 1$$

- Principal specifies $C_t$ and $k_t$ given history of returns

Based on joint work with Sebastian Di Tella (Stanford GSB)
Incentives to not steal

Optimal contract involves no stealing (stealing is inefficient) and no savings (without loss of generality)

Sensitivity of agent’s expected payoff to returns

\[ dW_t = r(W_t - \frac{C_t}{1-\gamma})dt + r\Delta_t (dR_t - (r + \alpha)dt) \]

Incentive constraint:

\[ \Delta_t \geq C_t^{-\gamma}\phi k_t \]

Remark: \( C_t \) affects incentives but depends on the precautionary motive
Incentives to not save

- Agent has no incentives to save when marginal utility
  
  $$C_t^{-\gamma}$$

  is a supermartingale (constant in expectation – or decreasing)

- Conditions on $W_t$ and $C_t$ rule out profitable deviations with stealing or saving (first-order approach)
- What about double deviations? Steal, save, consume later. We have to keep our fingers crossed…
Properties of solution

• Control problem, 2 state variables: $W$ and $C$
• Ratio $C/X$, where

$$X_t = \frac{1}{((1 - \gamma)W_t)^{1-\gamma}}$$

is “utility in dollars” reflects the agent’s precautionary motive.
  – $C/X$ goes down when precautionary motive increases
• Given $W_0$, the principal sets $C_0/X_0$ to maximize profit
• But $C_t/X_t > C_0/X_0$ for $t > 0$: principal distorts the contract to safety to reduce precautionary motive and improve incentives ex-ante
• $C_t/X_t$ goes up after bad outcomes
  – contract gets safer after bad outcomes – reduces precautionary motive
Verifying agent’s incentives

• We solved the principal’s problem subject to less than full set of incentive constraints by the agents (deviations with only “stealing” and savings)
• Fingers crossed – we are done only if agent does not want to deviate in any other way
• We can verify if we can find an upper bound on the agent’s deviation value function $U(W, C, h)$ such that

\[ U(W, C, h) = W \]
Verifying agent’s incentives

• We solved the principal’s problem subject to less than full set of incentive constraints by the agents (deviations with only “stealing” and savings)

• Fingers crossed – we are done only if agent does not want to deviate in any other way

• Verification:

\[
U(W, C, h) = \left(1 + \frac{hC^{-\gamma}}{(1 - \gamma)W}\right)^{1-\gamma} W
\]

is an upper bound of C/X goes up (contract gets safer) after bad performance.

• Intuition: if precautionary motive is reduced after bad performance, stealing and saving is unattractive
Some general properties

- Settings with hidden savings, private info about fundamentals, long-term consequences of actions:
- Optimal contract has distortions – principal commits to manage the agent’s future “information rents” to improve incentives ex-ante
- No reversibility – bad early performance cannot be undone by future good performance and vice versa
Some general properties

• Harder questions: need to add more state variables to summarize the agent’s incentives
  – Agent’s effort affects output now and in the future
  – Agent privately observes persistent shocks to output
  – Hidden savings

• Optimal contract has distortions – principal commits to certain ex-post inefficiencies to improve incentives ex-ante

• These distortions are history-dependent – irreversibility – bad early performance cannot be undone by future good performance
Conclusions

- Continuous-time methods offer huge potential to analyze dynamic agency models, which are common in economics.
- A lot of progress in the last decade, especially in corporate finance, but also in other fields.
- Three potential areas for future fruitful research:
  1. Effective numerical methods for solving control problems.
  2. Complexity of problems with persistent private information (like hidden savings) – when does the first-order approach work in general, and what to do when it does not work.
  3. Embedding models of dynamic contracts in broader macro settings – determination of interest rates, investment, business cycles, resource constraints.