Galileo's Use of Mathematics in its Historical Context

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1 Introduction

I will analyze the mathematical language used by Galileo Galilei (1564 - 1642) in his last treatise *Discorsi e dimostrazioni matematiche intorno a due nuove scienze*[1], and compare it to the trends of early modern mathematics of his times, namely, the advances in algebra and the development of a new notation for arithmetic and algebraic operations. I will attempt to explain the differences between Galileo's approach and how mathematics was evolving during his lifetime.

Studying the mathematical language employed by Galileo is essential to understand his theories, as one cannot separate the scientific results from the language used to express them. As shown by Jürgen Renn, for example in the case of arithmetic, the development of mathematical language and symbolic representation in a field has profound consequences for the field itself. It shapes how the subject is thought of, the entities studied, the allowed operations, the context in which the results are applied, and the level of abstraction reached[2]. In arithmetic, the development of context-dependent symbols for counting specific objects led, through iterative abstractions, to the birth of arithmetic operations valid for any object and the concept of number. In a physical theory, the mathematical language is also linked to its interpretation. A classical example is mechanics: in its Newtonian formulation, the state of the system at one instant determines the state immediately after, leading to a *causal* interpretation. Rewriting the theory using the principle of minimum effect, the evolution is given by the path that, through the history of the system, minimizes certain quantities, leading to a *teleological* interpretation of the same theory[3]. Therefore, studying the mathematical language of Galileo allows us to understand his theories in the way he formulated them.

I will begin by briefly analyzing how mathematics was developing during Galileo's lifetime. I will focus on the advances in algebra that had started in the Renaissance, and on how various mathematicians constructed a language to express their algebraic operations. I will then study the language of a particular result of Galileo concerning the strength of materials. Given the number of proofs in the treatise, choosing one offers only a partial picture of Galileo's mathematics, but it will be enough to identify the most important characteristic: the exclusive use of a geometric language. In the conclusion, I will comment on what can be learned from our analysis. There are important differences between Galileo's approach and the trends in mathematics that surrounded him, and it is important to explain them.

[1]In English, *Discourses and Mathematical Demonstrations Relating to Two New Sciences*, or in short *Two New Sciences*.
[2]Renn, 2022, pp. 87–96.
[3]von Neumann, 1963, pp. 477–490.

2 Early Modern Mathematics

The early modern period saw profound changes in the field of mathematics. Thanks to translations from Greek and Arabic, the works of Greek geometers, especially Archimedes, were rediscovered in Europe and became standard texts[4]. Galileo was well versed in Euclidean geometry, and Archimedes was an important author often referred to in his works. But the most important trend of that period was the development of algebra. This had its roots in the works of Leonardo da Pisa, better known as Fibonacci (1180 - 1250), who facilitated the transmission of Hindu-Arabic numerals and algebraic techniques from the Muslim world to Europe, and the friar Luca Pacioli (1445 - 1514), who contributed to the development of an algebraic notation[5].

During the 16th century, the solution for the cubic equation was found, thanks to the efforts of mathematicians Scipione del Ferro (1456 - 1526) and Niccolò Tartaglia (ca. 1499 - 1557)[6]. In studying the roots of cubic equations, Tartaglia and subsequent mathematicians found some problematic instances: equations that were known to have real roots required the use of square roots of negative quantities. In 1545 Girolamo Cardano (1501 - 1576) published his *Ars magna sive de regulis algebraicis*, in which he divulged the solutions of the cubic found by Tartaglia, and the solutions of the quartic equation found by Luigi Ferrari (1526 - 1565)[7]. Cardano had to deal with roots of negative expressions, which he described as being "so subtle as it is useless[8]".

In his treatise *L'Algebra*, Rafael Bombelli (ca. 1526 - 1573) made further use of complex numbers to solve different equations. Bombelli defined the imaginary unit $+\sqrt{-1}$ by its multiplication rules, and called it "more sophistic than real[9]," but he nonetheless argued for its use in solving mathematical problems[10]. During Galileo's life, algebra evolved into a field of mathematics completely different from the geometric approach of Euclid and Archimedes, which was still seen as the epitome of mathematical rigor. François Viète (1540 - 1603) was aware of this and in his *In artem analyticem isagoge* claimed to have created a new science, combining the rigor of geometry with the problem-solving power of algebra[11]. Viète created a literal calculus for mathematics, extending the notation used by his predecessors. He used letters for both unknown and parameters, which enabled him to write equations and identities in general form and to perform operations on these formulas to prove new results or solve mathematical problems.

By 1637, we can see in René Descartes' *La Géométrie* how mathematics had profoundly changed. The symbolic language adopted was by then very similar to the one in use today, as can be seen in table 1. This book marked the birth of analytic geometry, with its applications of algebra to geometry and geometry to algebra[12]. It is very improbable that Galileo read Descartes' book before the publication of *Two New Sciences*, and most of its contents were probably completed years before its publication in 1638[13]. I concluded with *La Géométrie* because it shows the magnitude of the changes undergone by mathematics during Galileo's lifetime, and he could not have been unaware of them. We know that Tartaglia, Cardano and Bombelli were among Galileo's readings[14], and it is highly probable that he was aware of the new trends of mathematics via his contacts with other mathematicians.

Author	Modern form	Author's form
Luca Pacioli (ca. 1447 - 1517)	$x^2 + x = 12$	1.ce.p̃.1.co.e q̃ le a 12
Girolamo Cardano (1501 - 1576)	$x^3 = 15x + 4$	1.$cu.aequalis$15.$rebus$ p̃.4
Rafael Bombelli (ca. 1526 - 1573)	$x^6 - 10x^3 + 16 = 0$	1.$\underline{6}$ m.10 $\underline{3}$ p̃.16 eguale a 0
François Viéte (1540 - 1603)	$x^3 + 3bx = 2c$	$Acubus + Bplano3inA\ aequari\ Zsolido2$
René Descartes (1596 - 1650)	$x^3 + px + q = 0$	$x^3 + px + q \propto 0$

Table 1: Evolution of mathematical notation up until the publication of *Two New Sciences*. The examples are taken from a similar table in (Isabella Grigoryevna Bashmakova, 2000, p. 78), some typos have been corrected.

[4]Uta C. Merzbach, 2011, pp. 235–236.
[5]See (Uta C. Merzbach, 2011, pp. 227–253), (Oaks, 2010, pp. 1–38), (Heeffer, 2010).
[6]See (Isabella Grigoryevna Bashmakova, 2000, pp. 67–69) and (Toscano and Sangalli, 2020).
[7]Isabella Grigoryevna Bashmakova, 2000, pp. 70–71.
[8]The original reads: "*[...] adeo est subtile, ut sit inutile.*" From (Cardano, 1545, p. 66).
[9]The original reads "*[...] più tosto sofistica che reale [...]*" From (Bombelli, 1572, p. 169).
[10]Isabella Grigoryevna Bashmakova, 2000, pp. 71–75.
[11]Isabella Grigoryevna Bashmakova, 2000, pp. 75–80.
[12]Uta C. Merzbach, 2011, pp. 309–320.
[13]Valleriani, 2010, p. 120.
[14]Favaro, 1887, pp. 48–52.

3 A Result from *Two New Sciences*

I will examine one of the results contained in *Two New Sciences*, proposition VIII of the Second Day of the treatise (concerning the strength of materials)[15]. This result is about the cantilever beam, a horizontal beam only supported at one end, as shown in figure 1. I chose this particular proof because it showcases the geometrical method used by Galileo, and allows us to compare it to the algebraic language which was developing in the early modern period. I will translate Galileo's geometrical propositions into modern mathematical language, in order to make the reasoning more understandable.

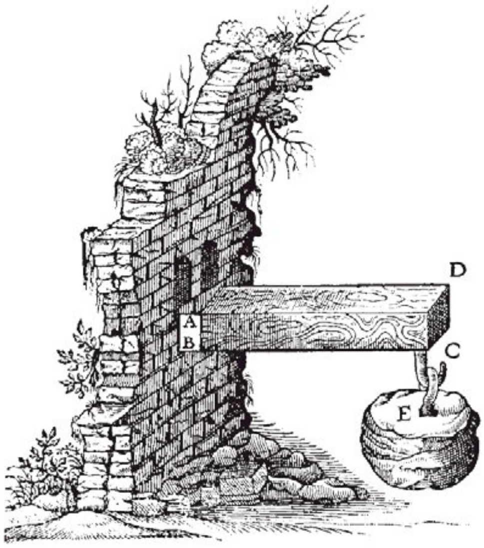

Figure 1: A cantilever beam, as found in *Two New Sciences*. Adapted from (Galileo Galilei, 1898, p. 157).

The goal of Galileo is the following:

> Given a cylinder or prism of the maximum length so that it does not break due to its own weight, and given a greater length, find the width of another cylinder or prism that, under the given length, is the only one and maximum that can resist under its own weight[16].

Galileo has already shown that in a beam, if one keeps the width constant and increases the length, after a certain point the weight of the beam will be enough to make it collapse. The goal now is, given a beam of length l and width h, and a new length $l' > l$, to find the corresponding width h' so that the new beam is such that l' is the maximum length it could have before collapsing. Galileo describes how to construct this new beam, providing the reader with figure 2 to help understanding his method.

> Let the cylinder BC be the maximum that can resist its own weight, and let DE be a length greater than AC: we need to find the width of the cylinder that under the length DE is the maximum that can resist under its own weight. Let I be the third proportional of the lengths DE, AC, and as DE to I, so let the diameter FD to the diameter BA, and build the cylinder FE; I say that this is the maximum and only one, among all the ones similar to it, that can resist to its own weight[17].

The cylinder BC then is the starting one, with length l (AC) and width h (AB), and the goal is to construct the cylinder of length l' (DE) such that it is the maximum that can support its own weight. Galileo constructs I, the third proportional of DE and AC. This is a geometrical approach which follows the definition of third proportional that can be found in proposition 11 of book VI of Euclid's Elements[18].

[15] I will not rewrite the original Italian version, it can be found at pages 166 and 167 of (Galileo Galilei, 1898). The translation is mine, but a reference can be (Galilei, 1989). Due to space constraints, I will also not analyze the proof of the result.

[16] Galileo Galilei, 1898, p. 166.

[17] Galileo Galilei, 1898, pp. 166–167.

[18] Heath, 1908, p. 214.

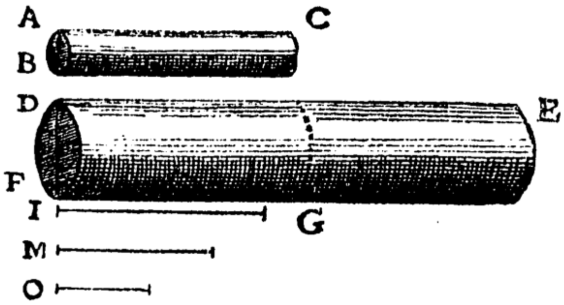

Figure 2: Galileo's picture to help the reader understand his proof. Adapted from (Galileo Galilei, 1898, p. 167).

In algebraic terms, the third proportional c of a and b is such that

$$a : b = b : c \quad ,$$

which means that

$$c = \frac{b^2}{a} \quad .$$

In our case then, I is such that its length i is

$$i = \frac{l^2}{l'} \quad .$$

Galileo now states that the required diameter h' (FD) is such that

$$l' : i = h' : h \quad ,$$

which means that

$$h' = h\frac{l'^2}{l^2} \quad .$$

This is the required result, a scaling law that shows how bodies become progressively weaker when scaled up keeping their proportions fixed. If we have a beam of length l and width h which can barely support its own weight, and we want to find a new beam of length $2l$ for which the same condition holds, we cannot just double the width, but we need to multiply it by 4, clearly breaking scale invariance[19]. The proof is a convoluted geometrical argument, that requires the construction of different lines in order to obtain the right exponents of known quantities without any algebraic manipulation, as seen in figure 2.

4 Analysis and Conclusion

The first feature one can notice in Galileo´s language is the absence of any algebraic approach or notation. Galileo only used geometrical constructions, even if simple rules of literal calculus would have allowed him to raise a known quantity to a given power or dividing it by another quantity, without constructing other lines with the required magnitude. It is important to identify possible reasons for only utilizing geometrical methods in order to better understand Galileo's science.

Galileo was trained in the classical tradition of geometry, as can be seen in his works and his references to authors such as Euclid and Archimedes, but his avoidance of modern methods cannot be attributed to ignorance of them, since he was acquainted with the works of Renaissance mathematicians and with the evolution of mathematics. A more believable explanation should take into consideration how algebra and symbolic analysis were seen in his time. The new methods were mostly considered heuristic tools for finding solutions of equations[20]. Geometry, on the other hand, proceeded from postulates to secure truths through demonstrations. The new branch of algebra was simply not yet perceived as providing

[19]Galileo's geometrical language is not easy to interpret: some historians of science incorrectly reported this result as $h' = h\sqrt[3]{\frac{l'}{l}}$ (Alessandro De Angelis, 2021, p. 120).

[20]Kaplan, 2018, p. 447.

the same kind of absolute truths as geometry. Even Newton, while using symbolic manipulations and the new calculus to achieve his results, preferred to publish his findings in a geometrical language[21]. Galileo's use of geometrical methods in his works reflects how the different branches of mathematics were perceived at the time.

Another element to keep in mind is that the new trends of mathematics were highly context-dependant. Like with the birth of arithmetic, where symbols and methods born in the context of counting objects became, through iterative abstractions, universal rules that could be applied to a general notion of "number", so the birth of the algebraic language must not be separated from its context[22]. The origins of these new problem-solving methods were rooted in bookkeeping, and then applied by Renaissance and early modern mathematicians to geometrical problems, which mostly consisted in finding the numerical magnitudes of given geometrical figures. Without anachronistically applying a modern mindset to analyze the question at hand, there are no obvious reasons why these methods should have explained truths pertaining to natural philosophy. These considerations show, once again, that it is impossible to separate a scientific theory from the language in which it was formulated.

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[21] Kaplan, 2018, pp. 447–448.
[22] Renn, 2022, pp. 88–96.