Gaussian Processes for Regression: Models, Algorithms, and Applications

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MIT

http://www.tamarabroderick.com/tutorials.html

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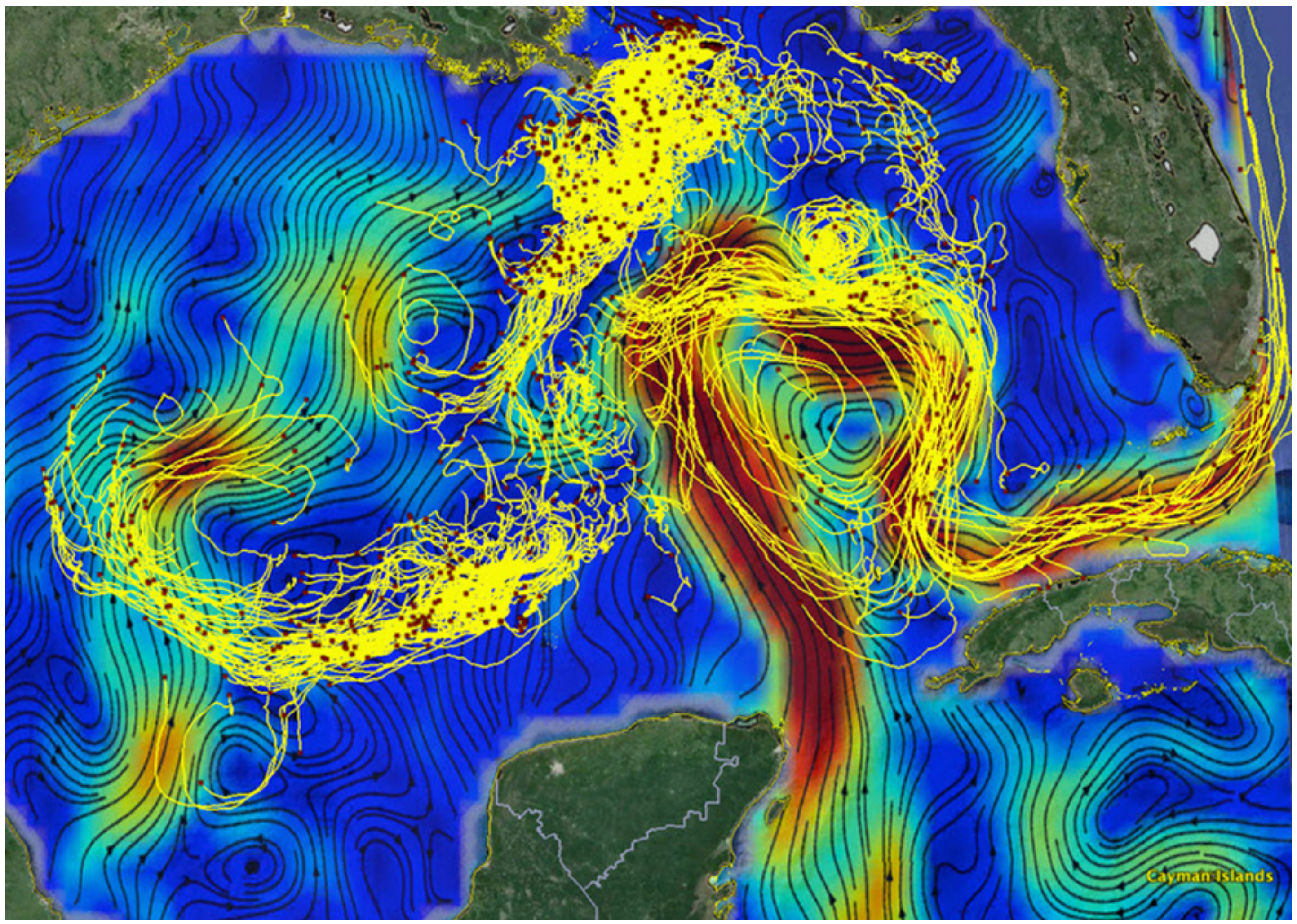

Example:

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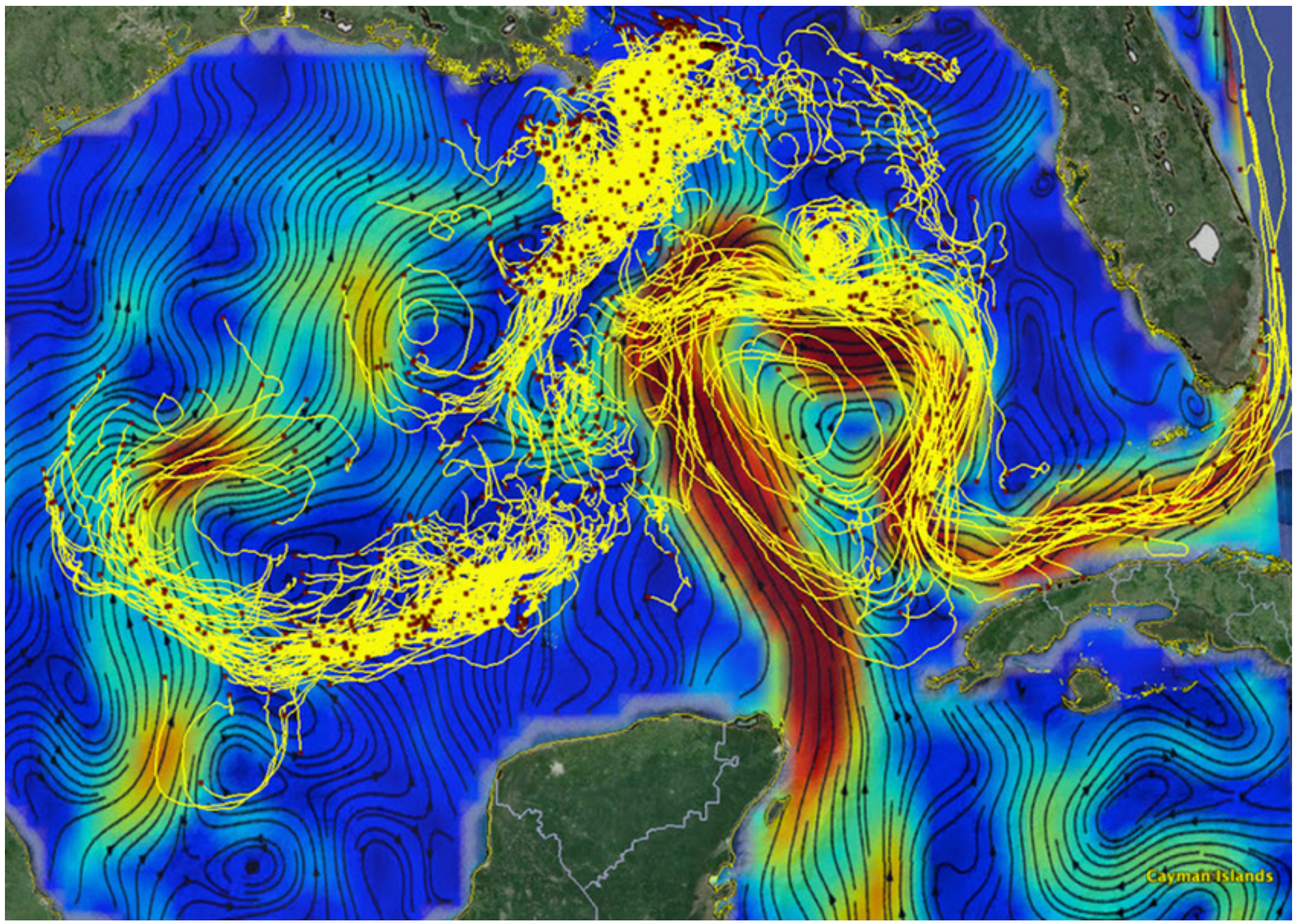

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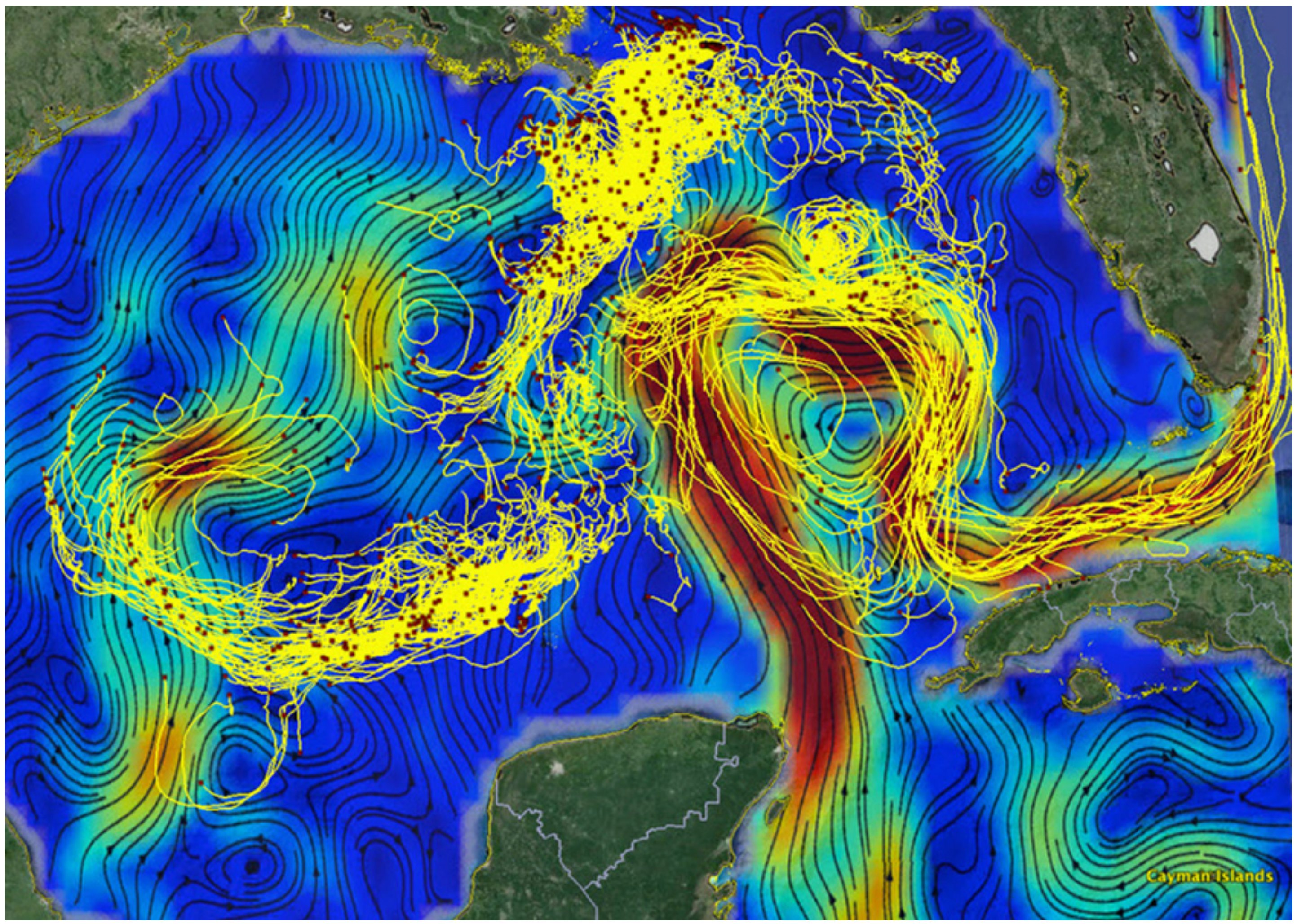

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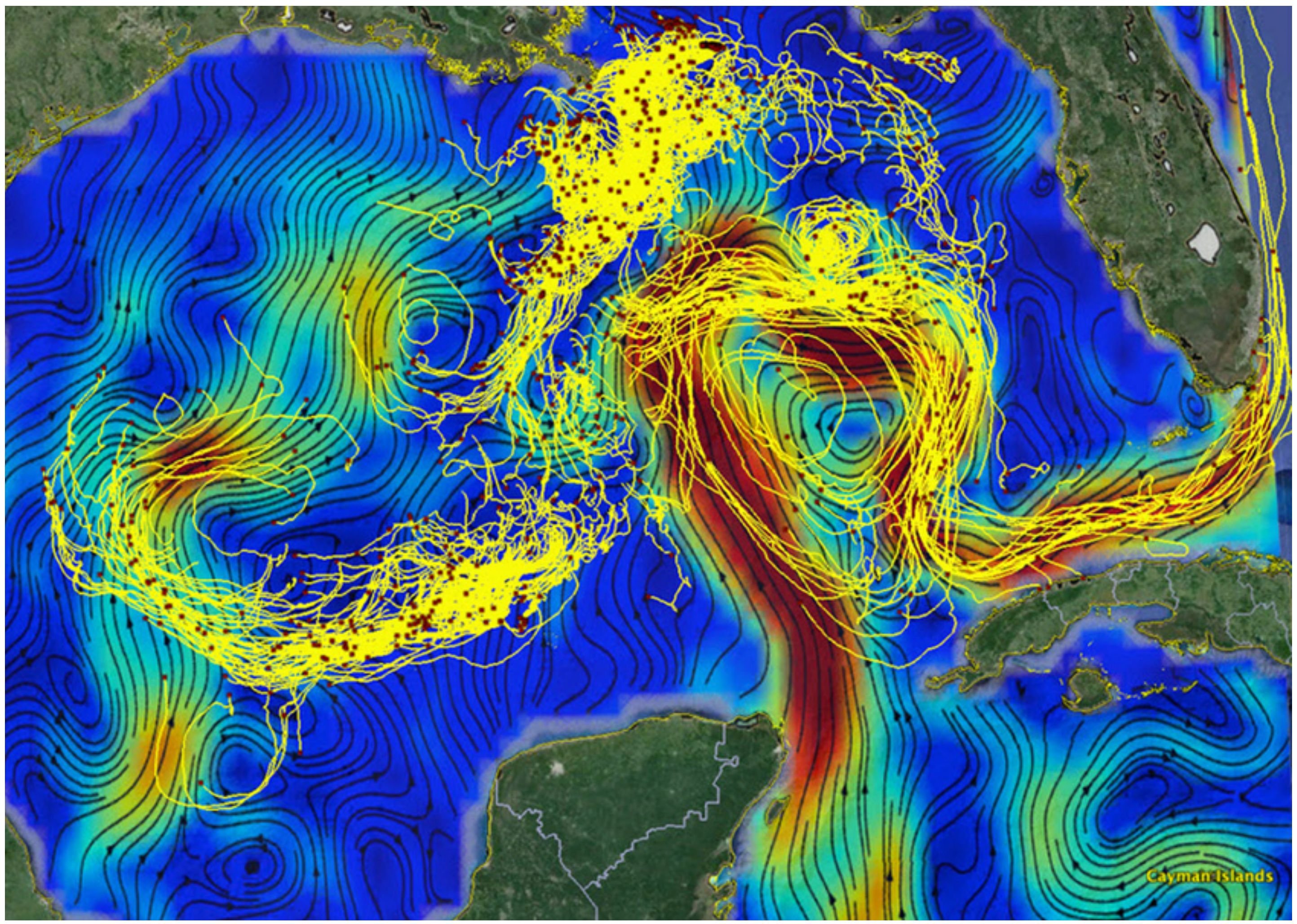

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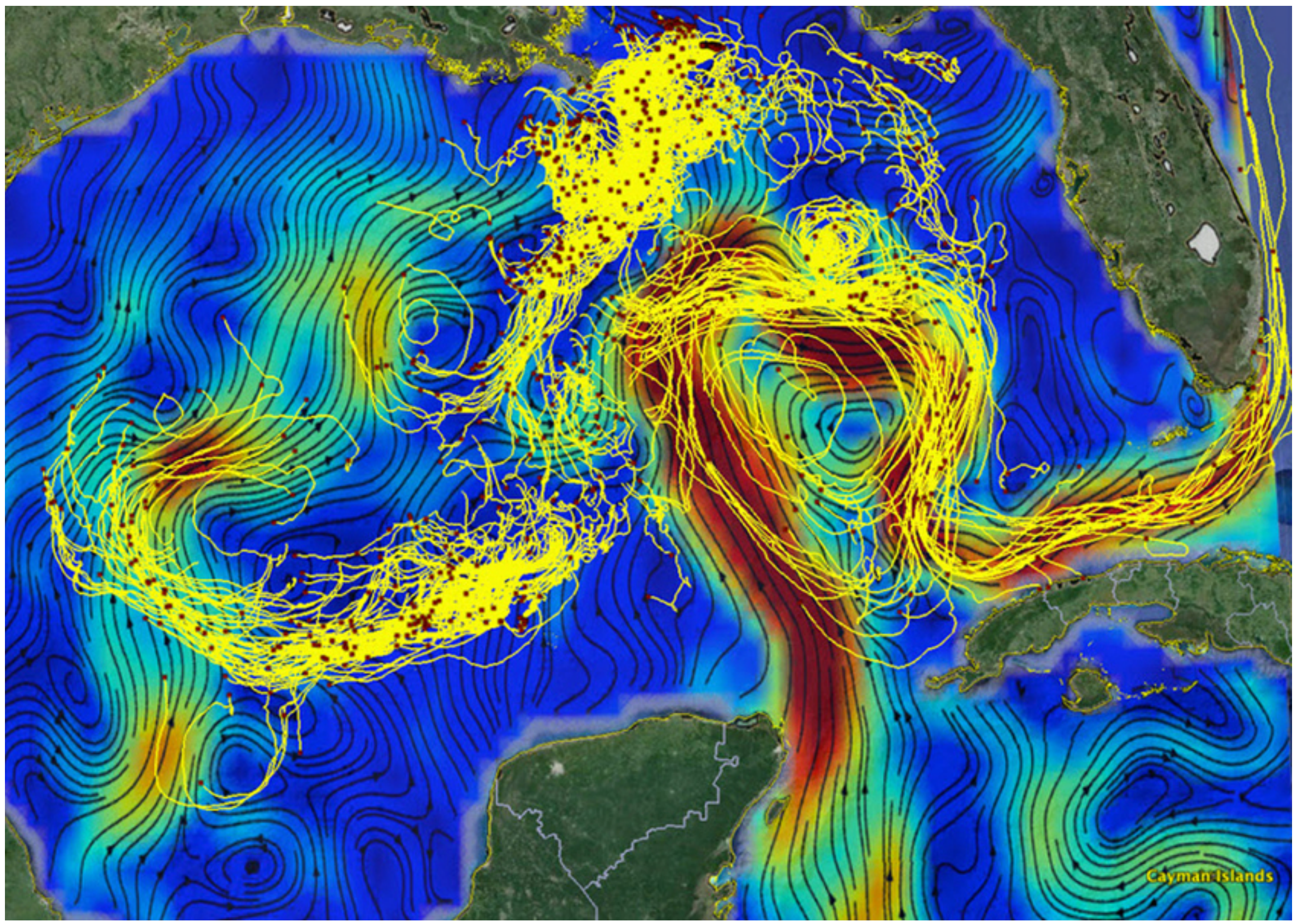

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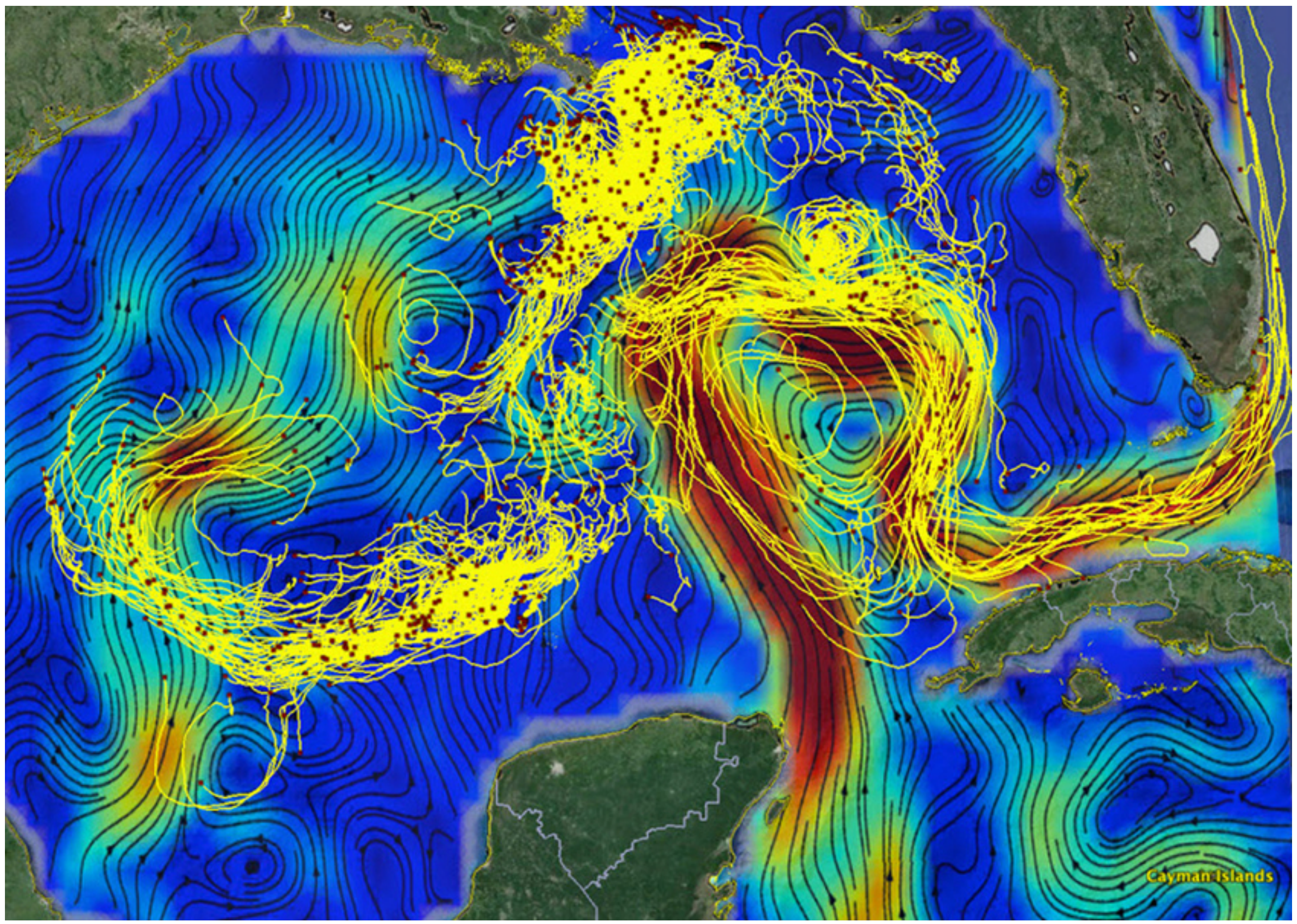

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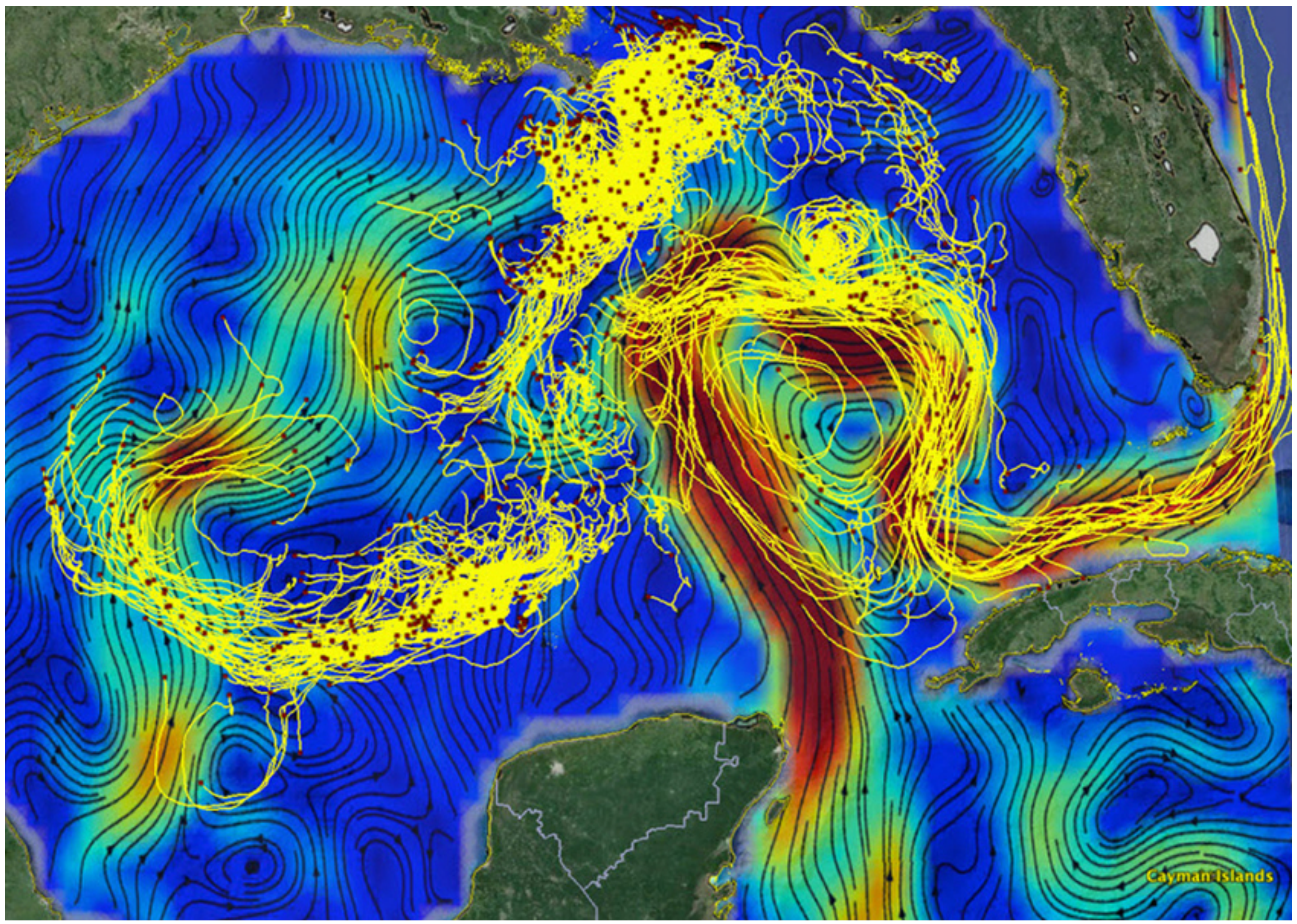

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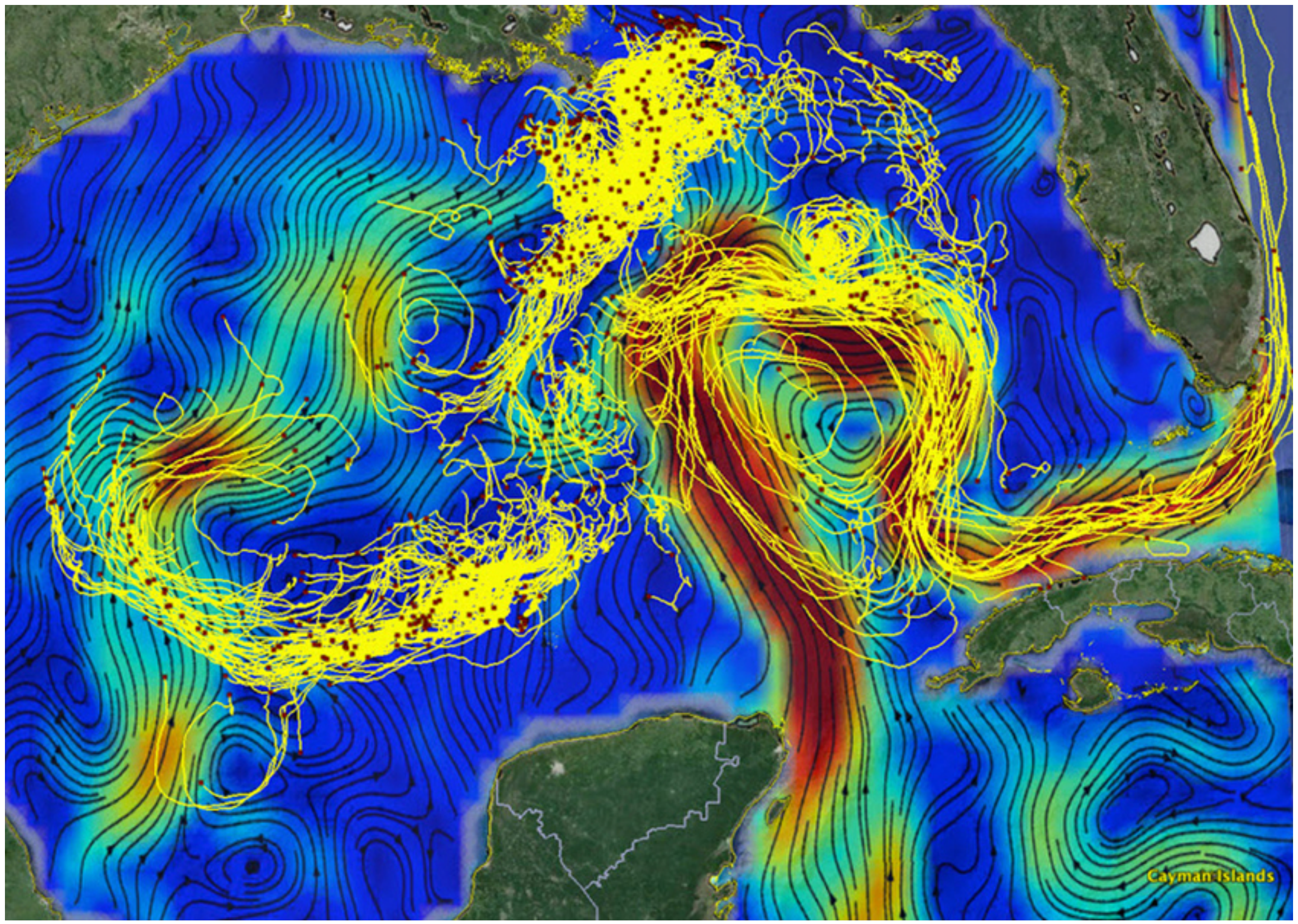

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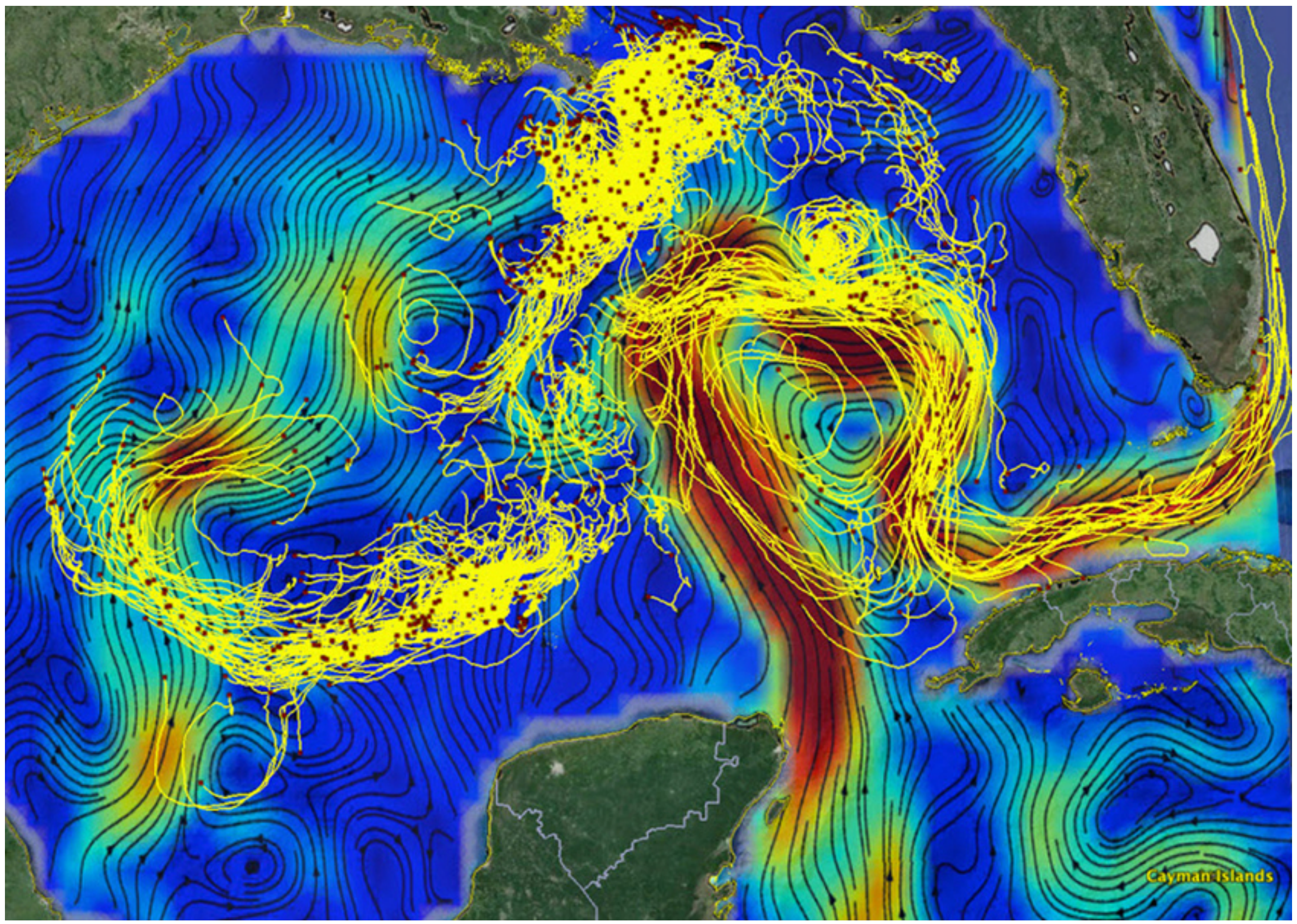

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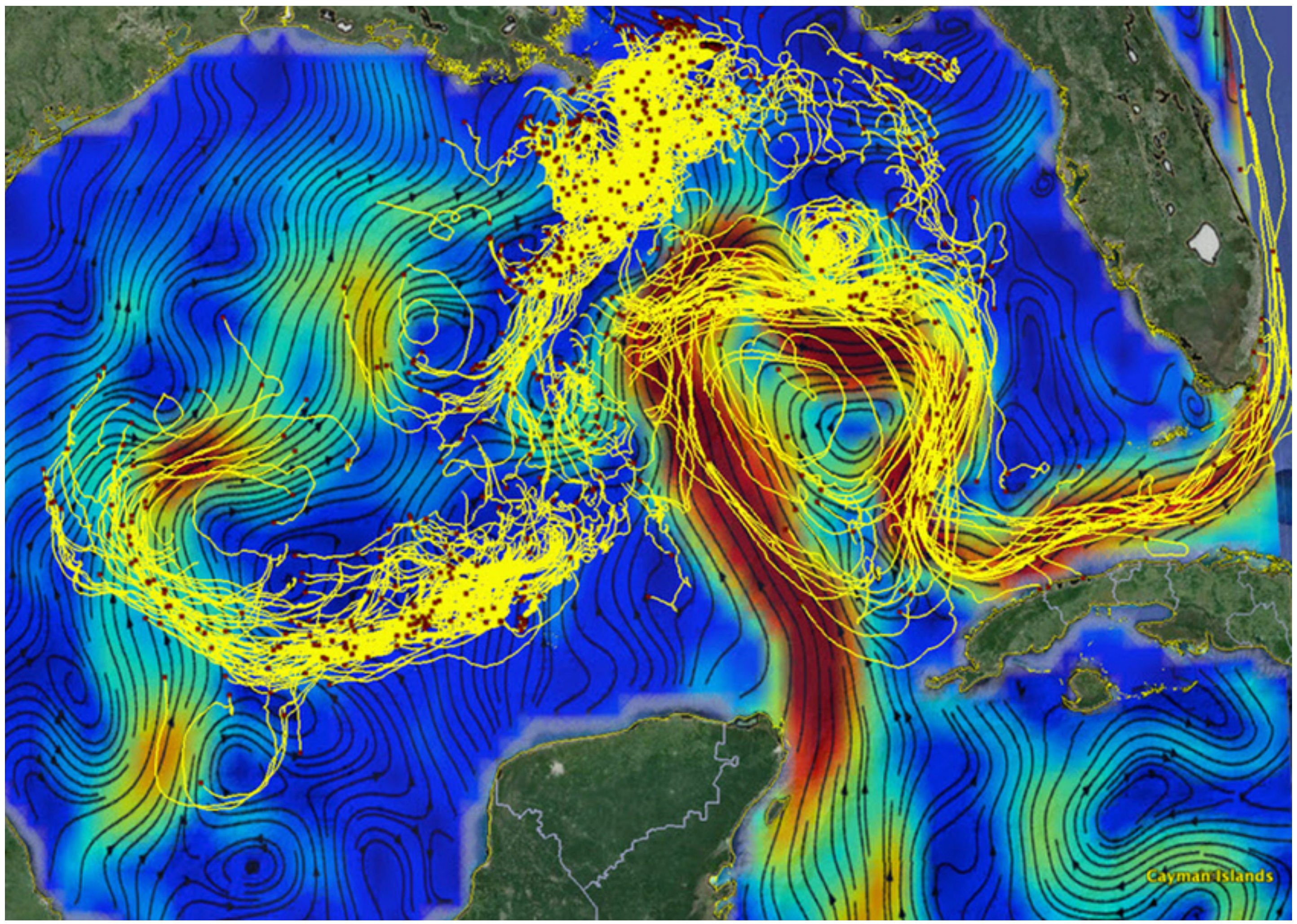

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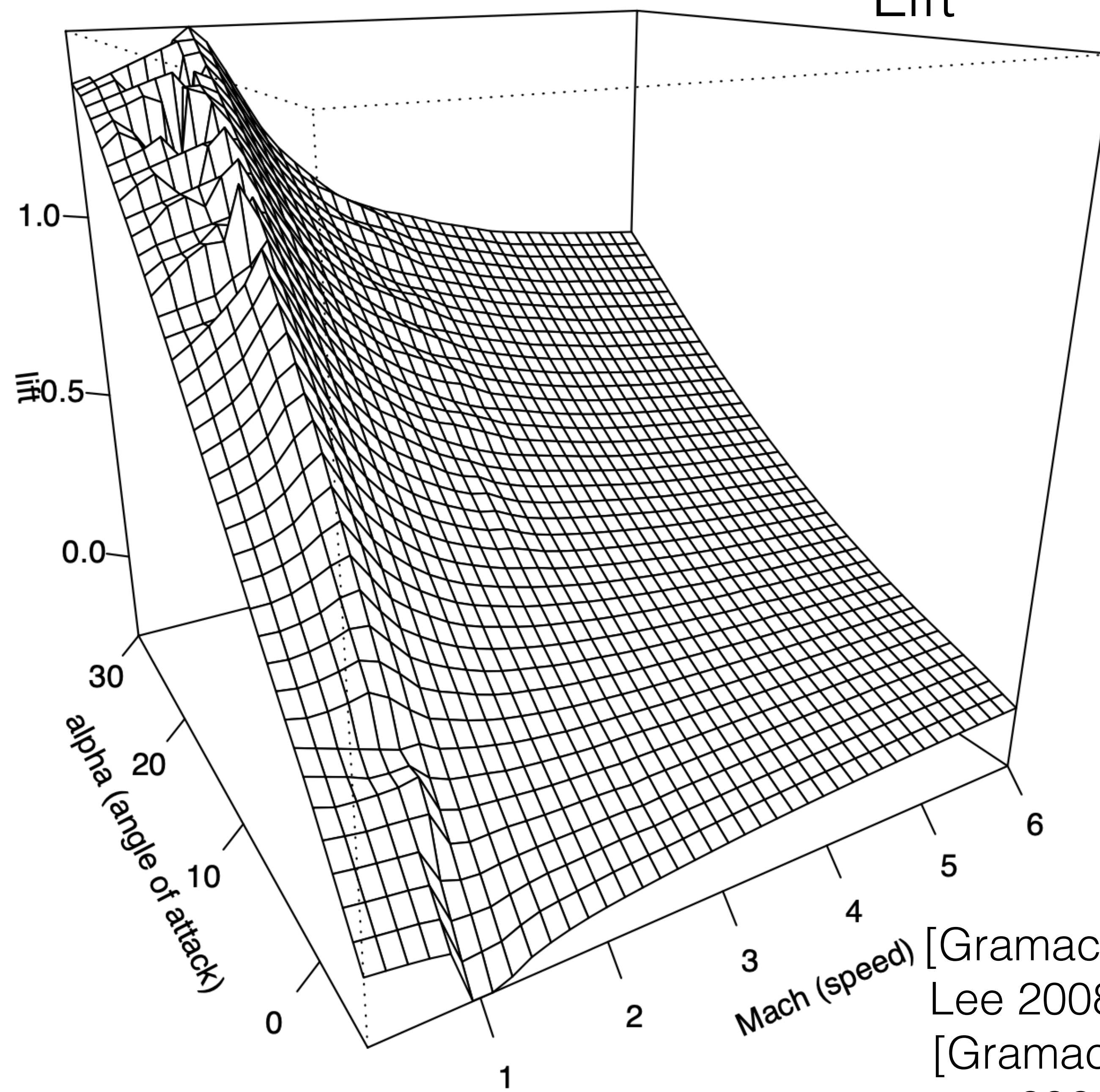

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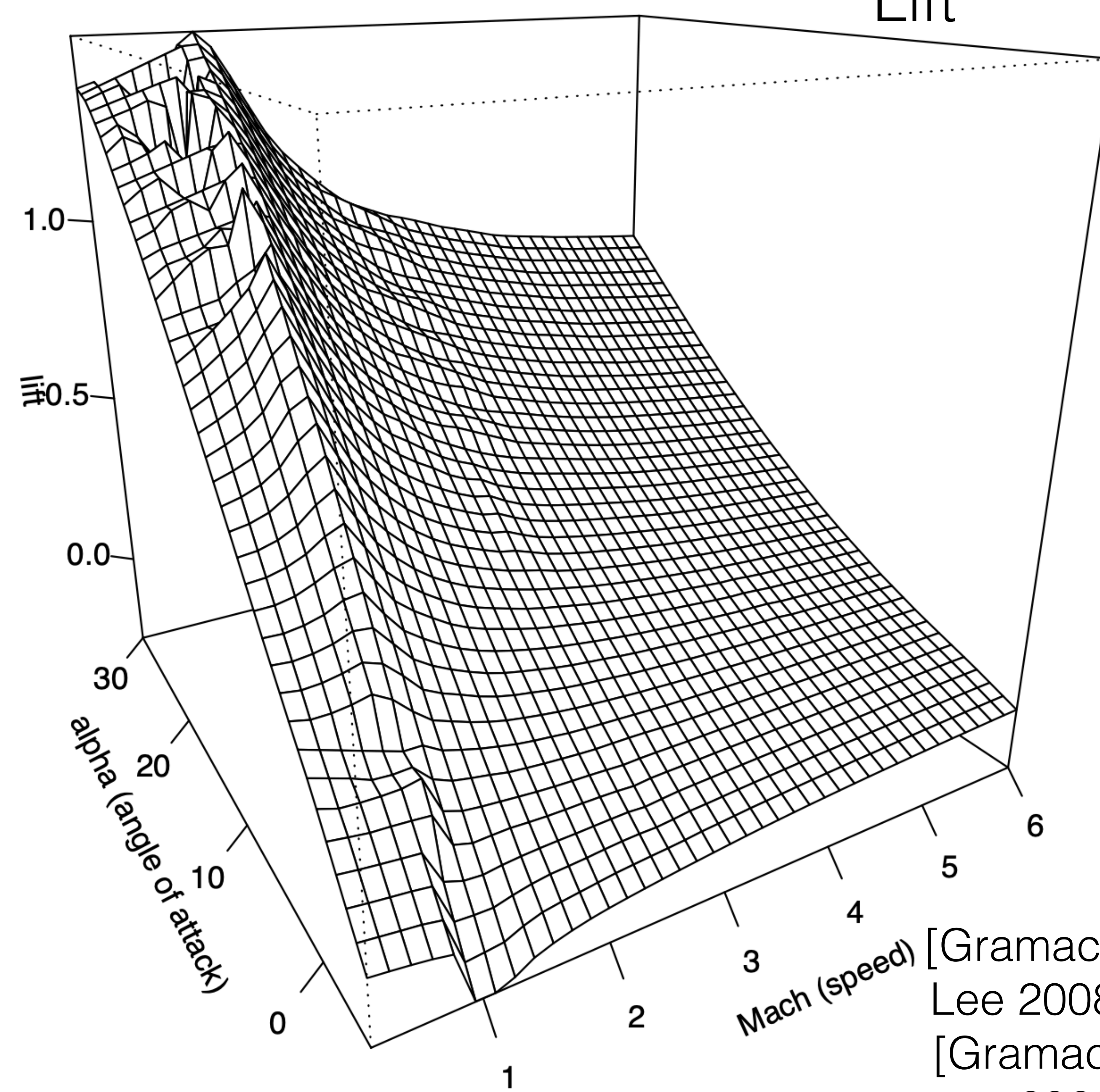

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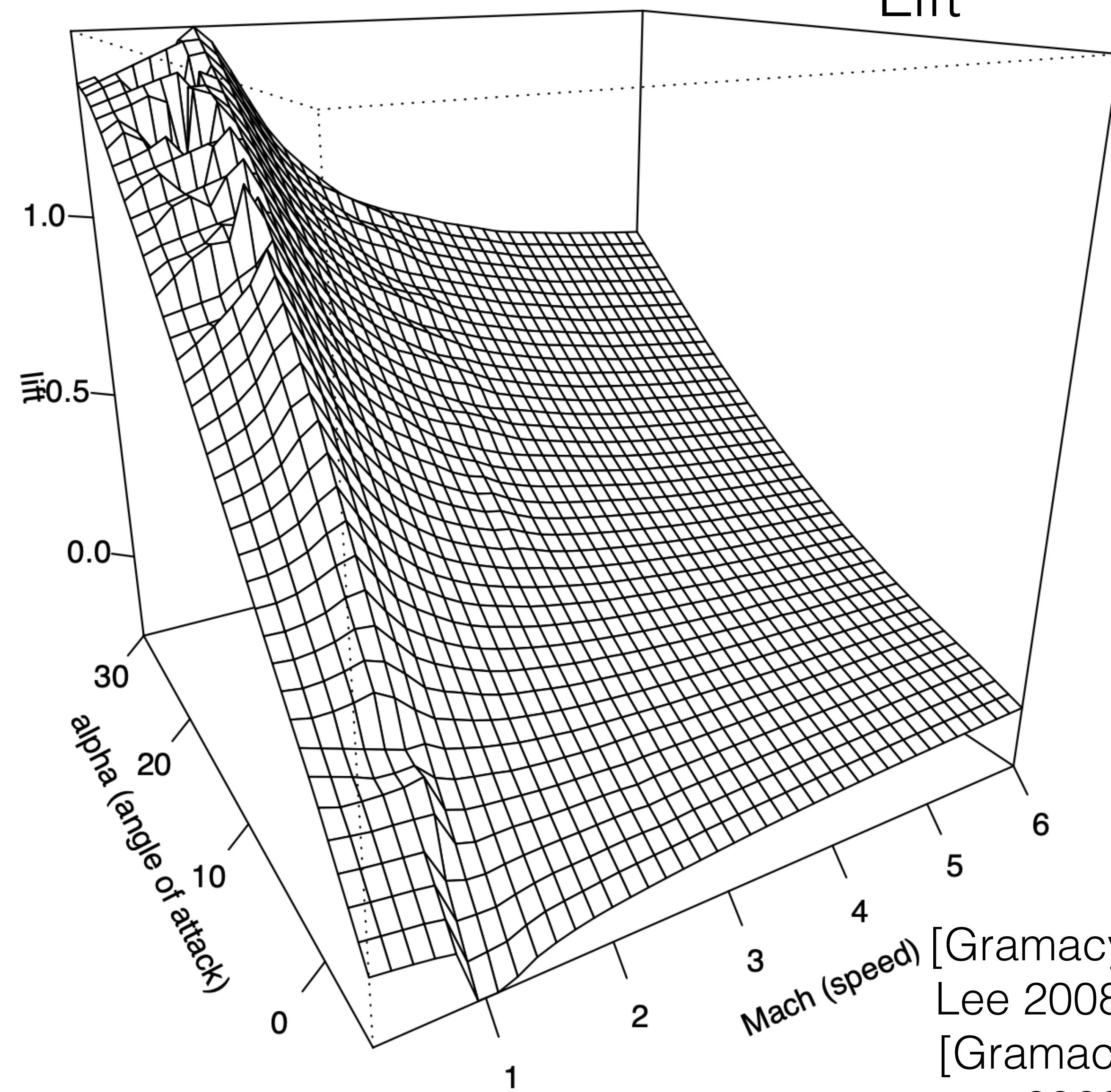

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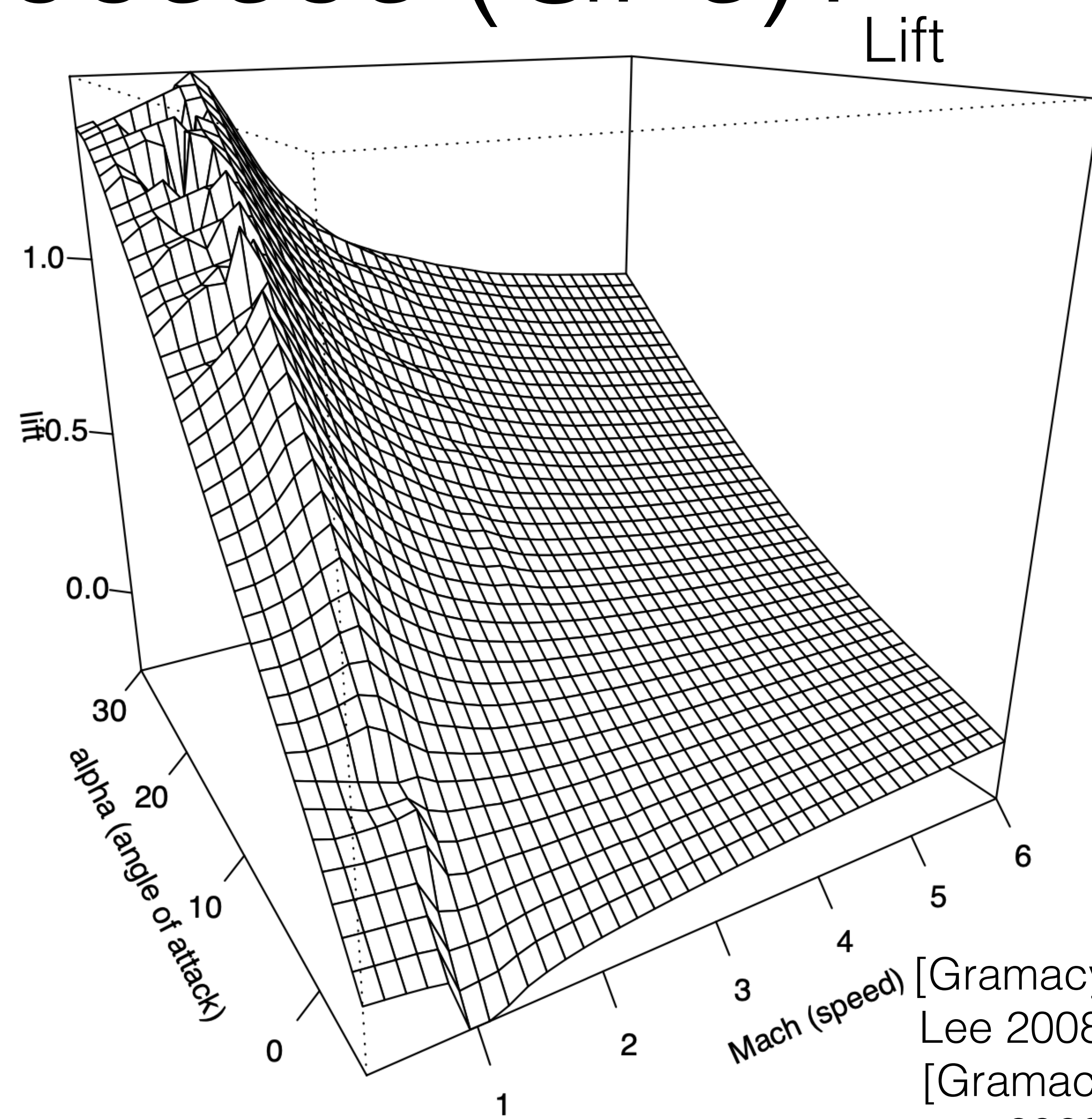

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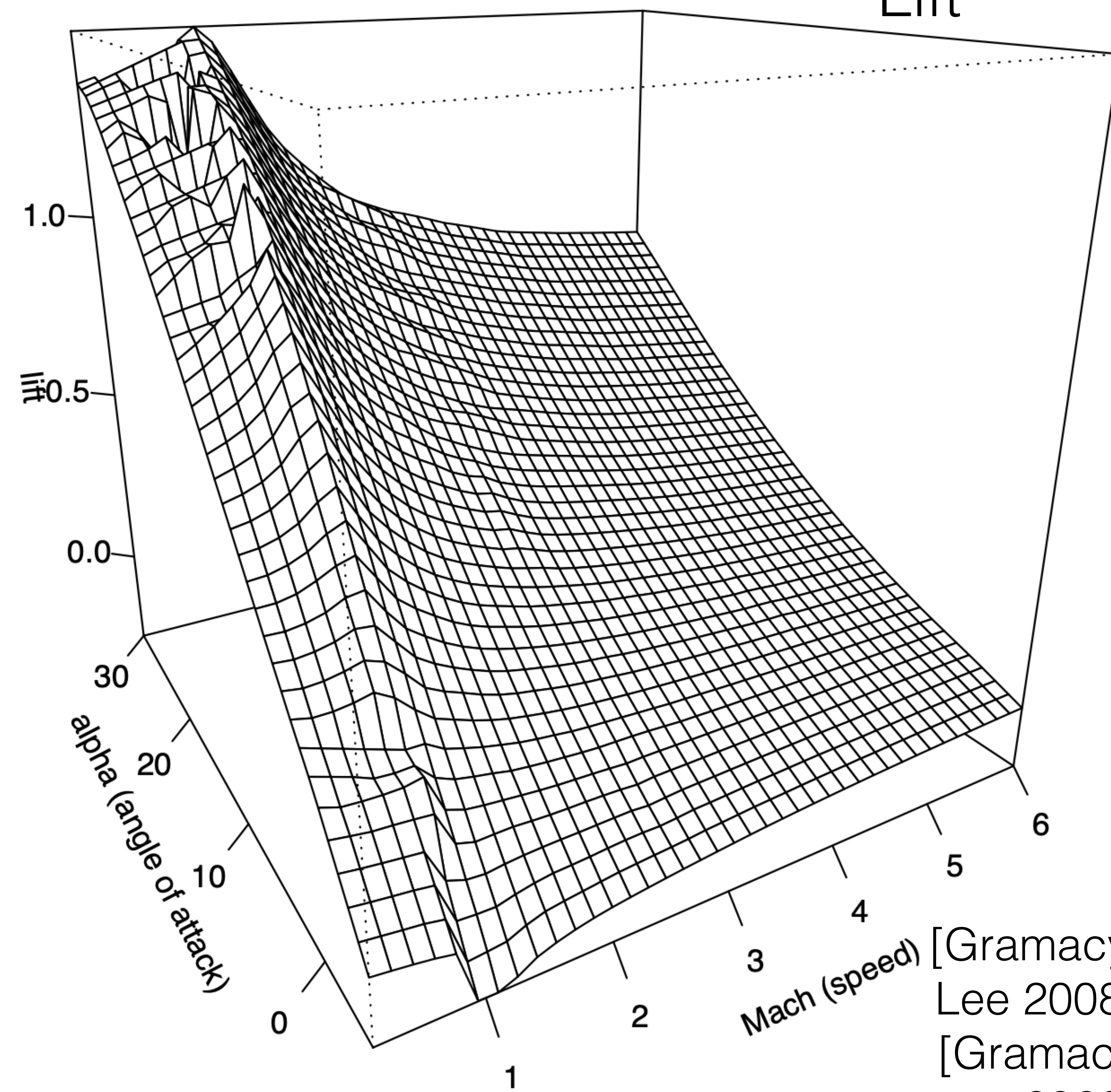

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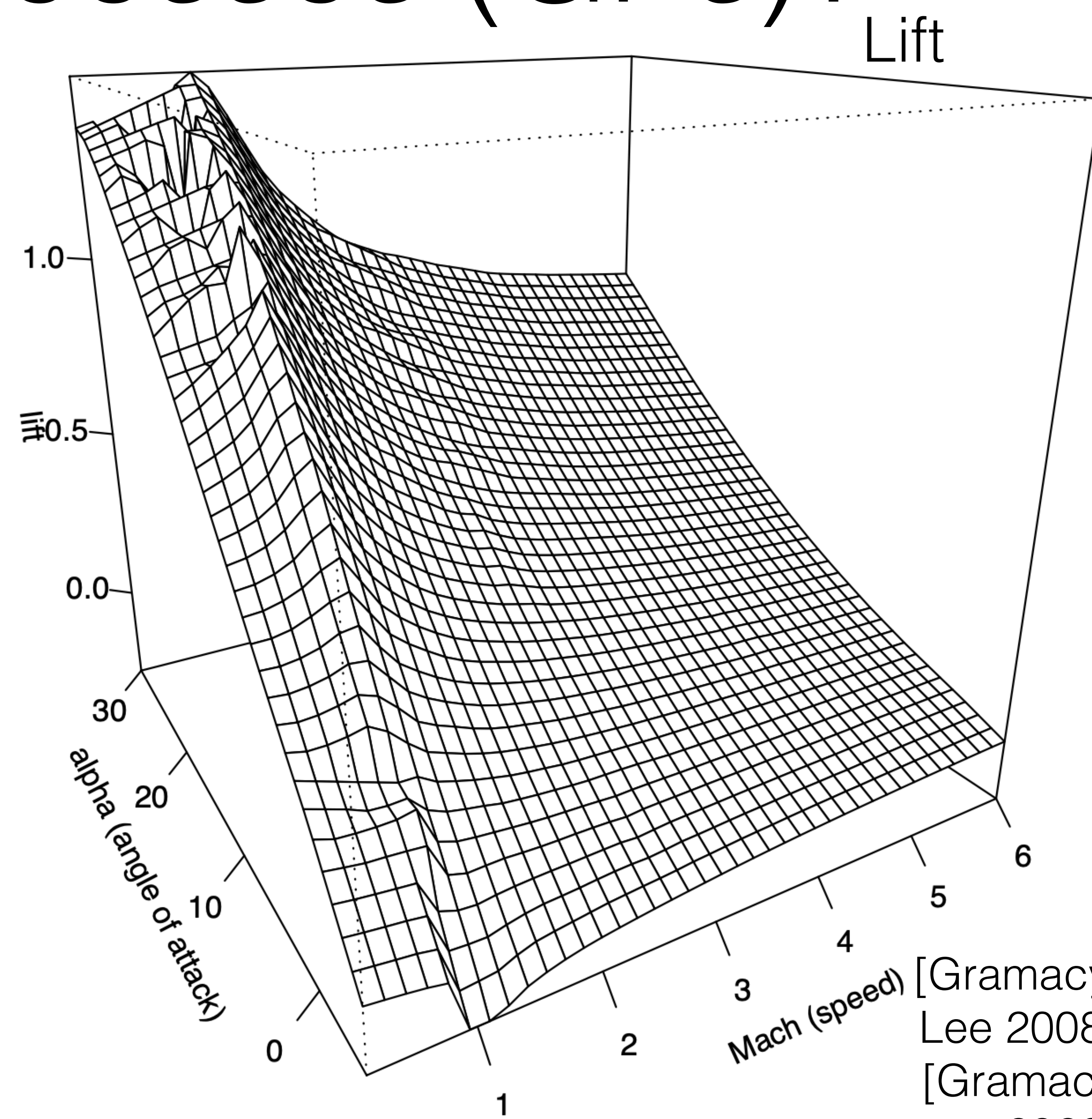

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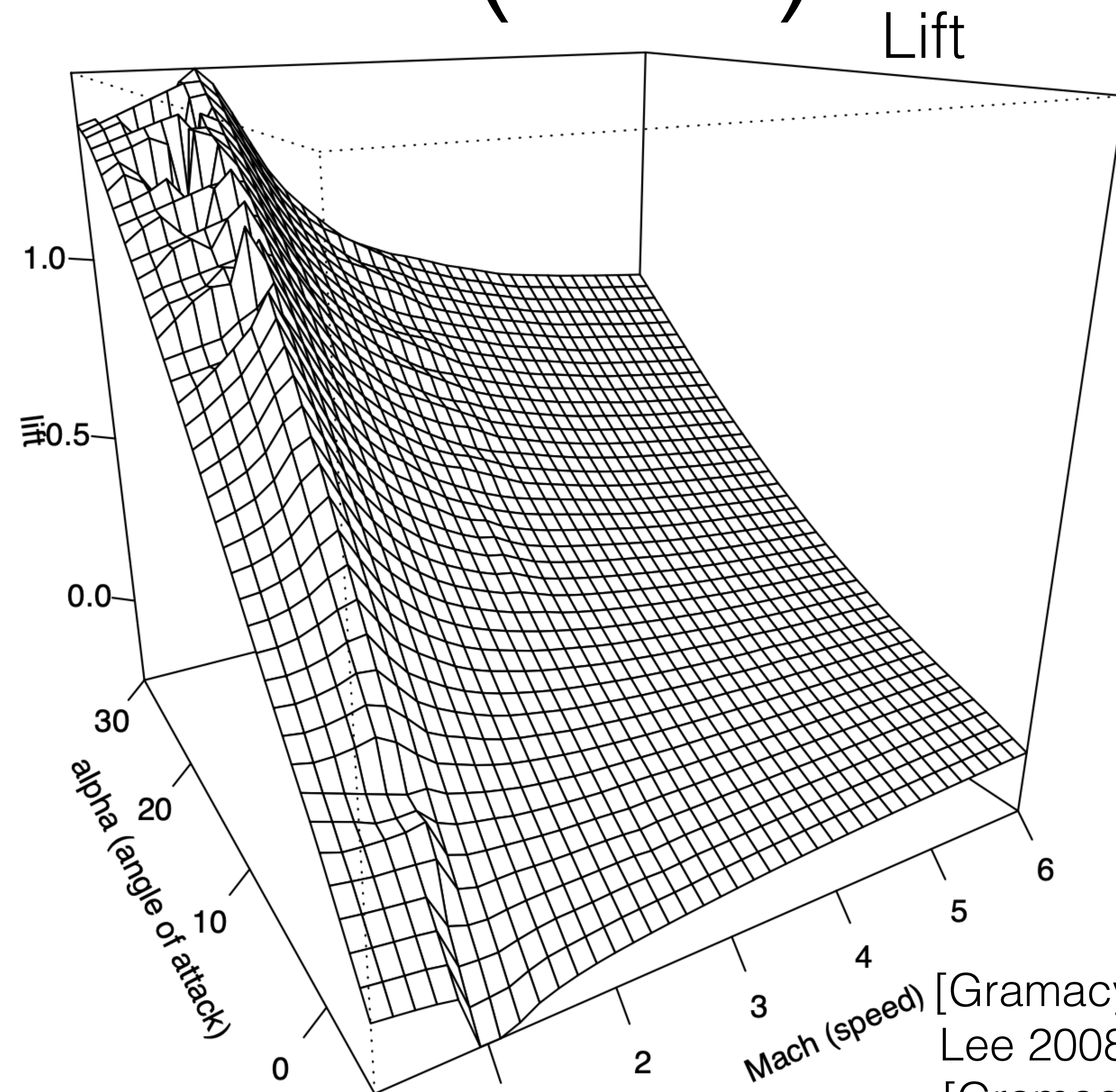

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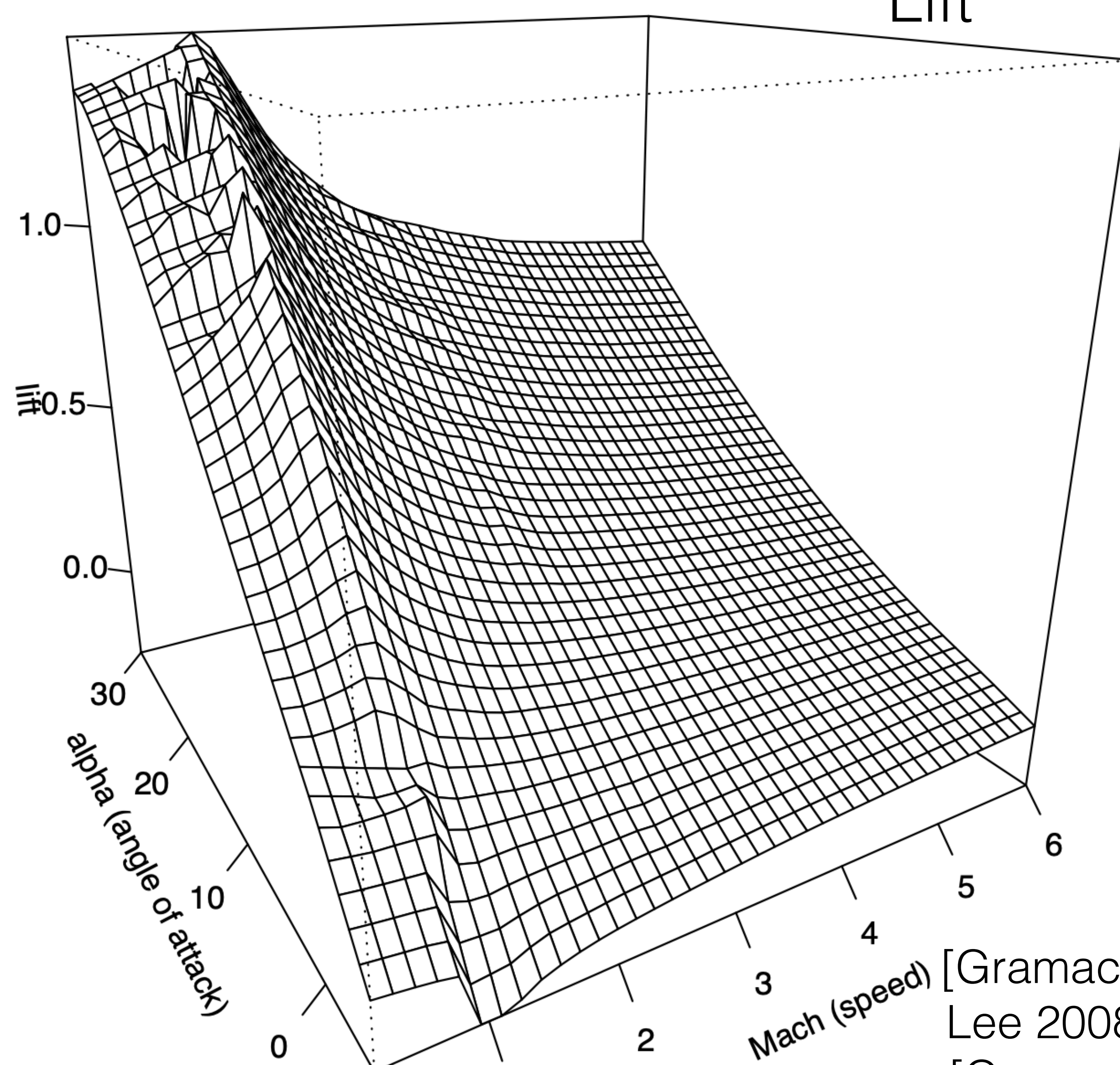

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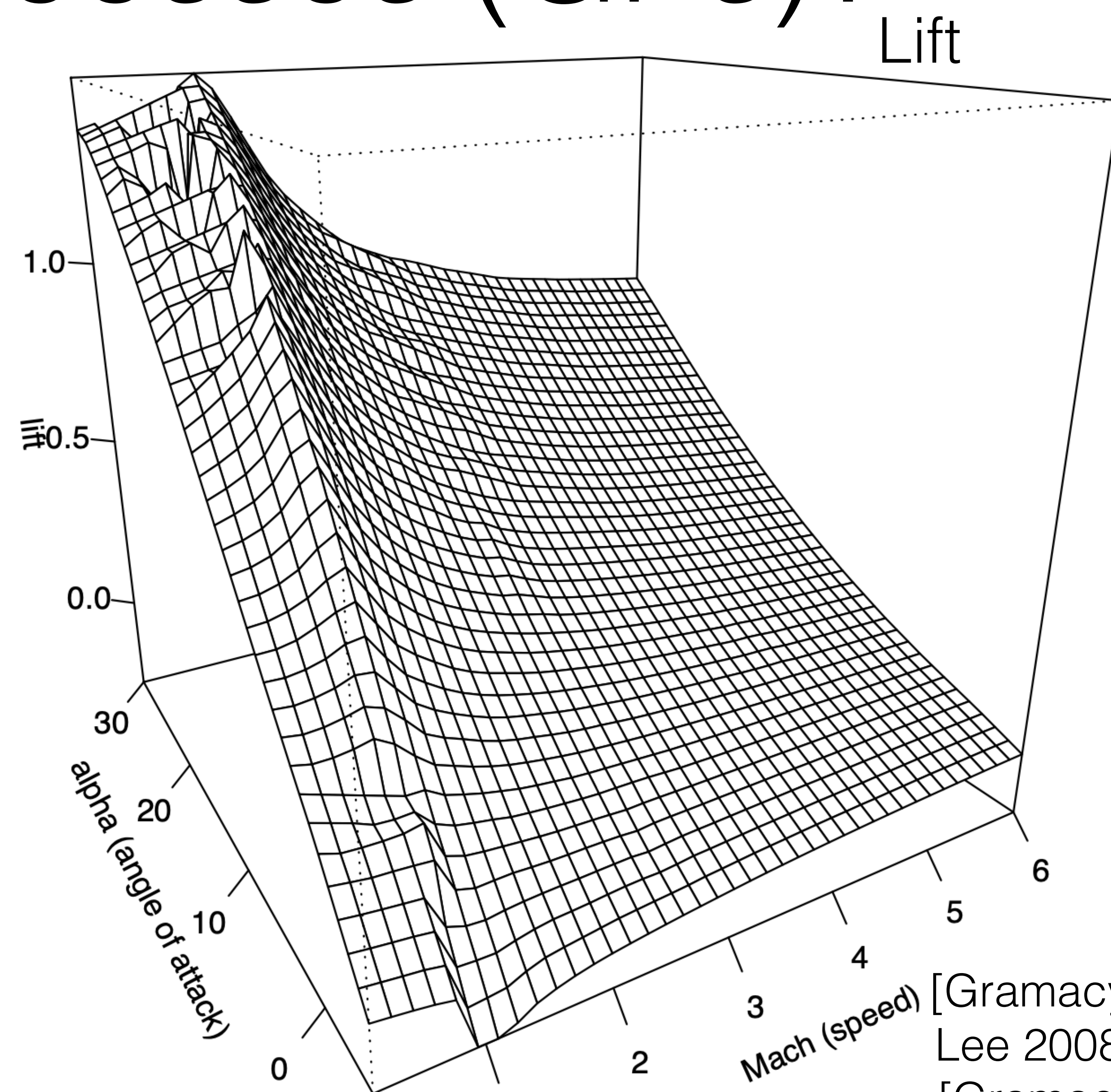

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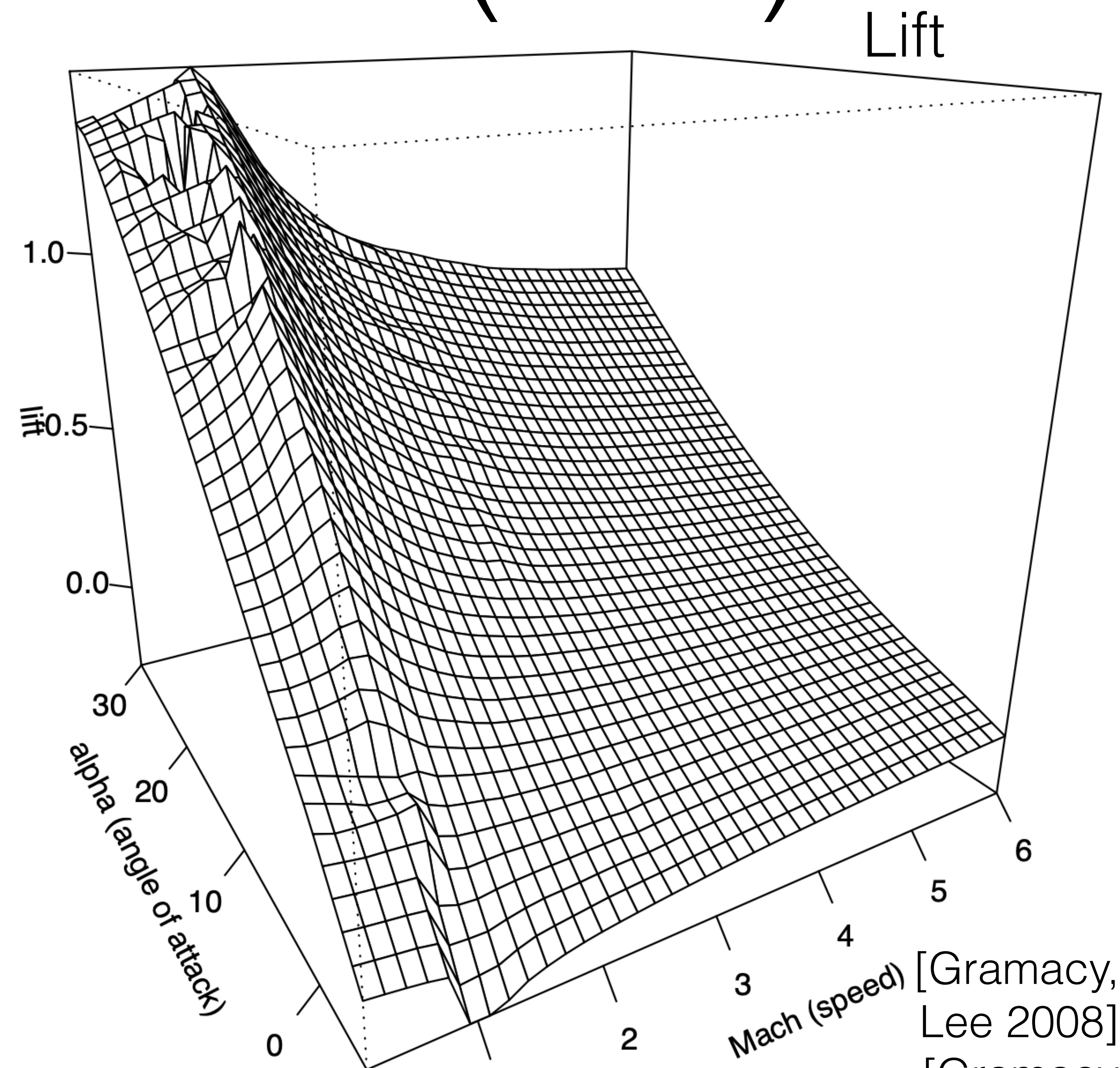

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One more example: learn (& optimize) performance in machine learning as a function of tuning parameters

[Snoek et al 2012, 2015; Garnett 2023]

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- Goal:
    - Learn the mechanism behind standard GPs to identify benefits and pitfalls

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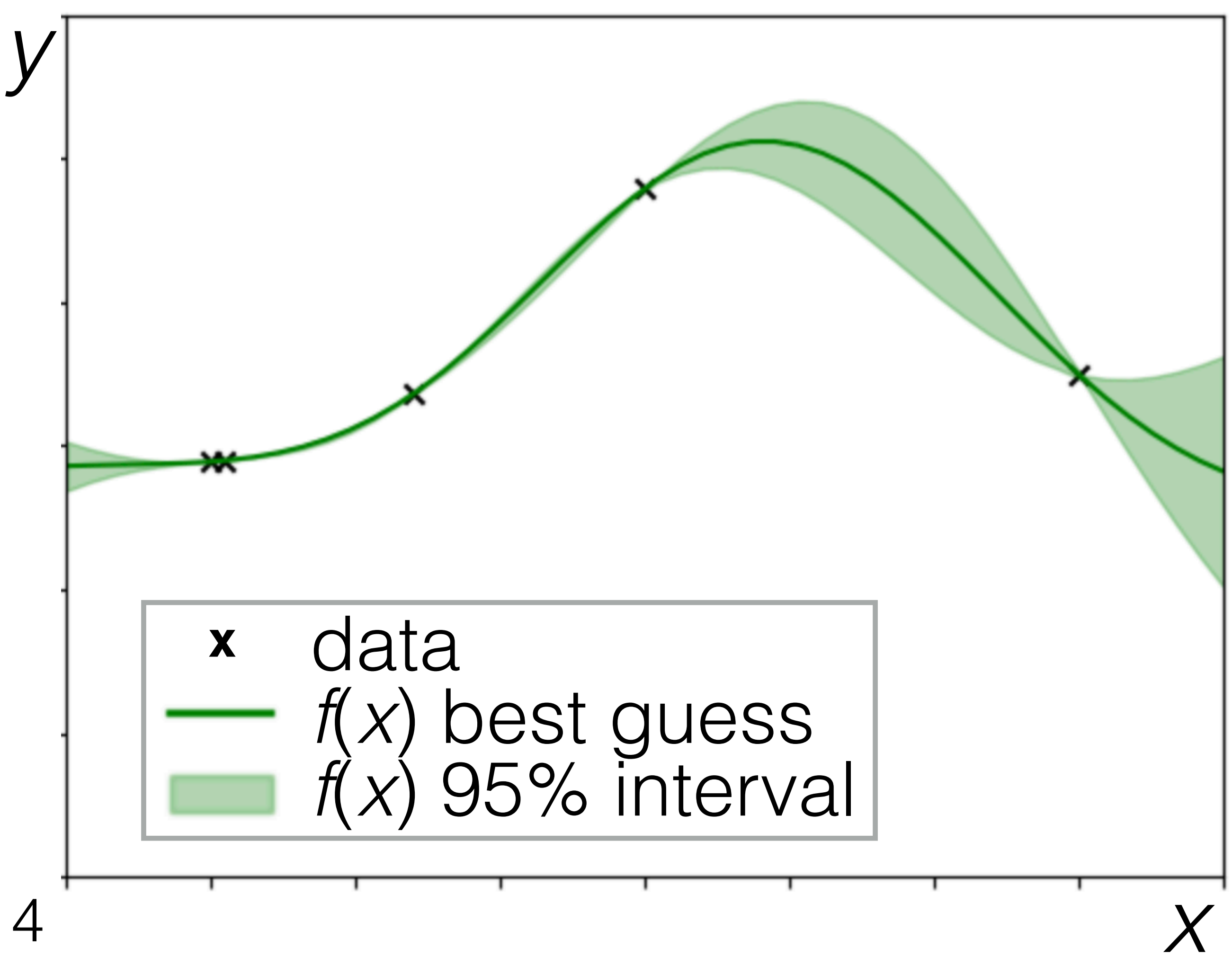

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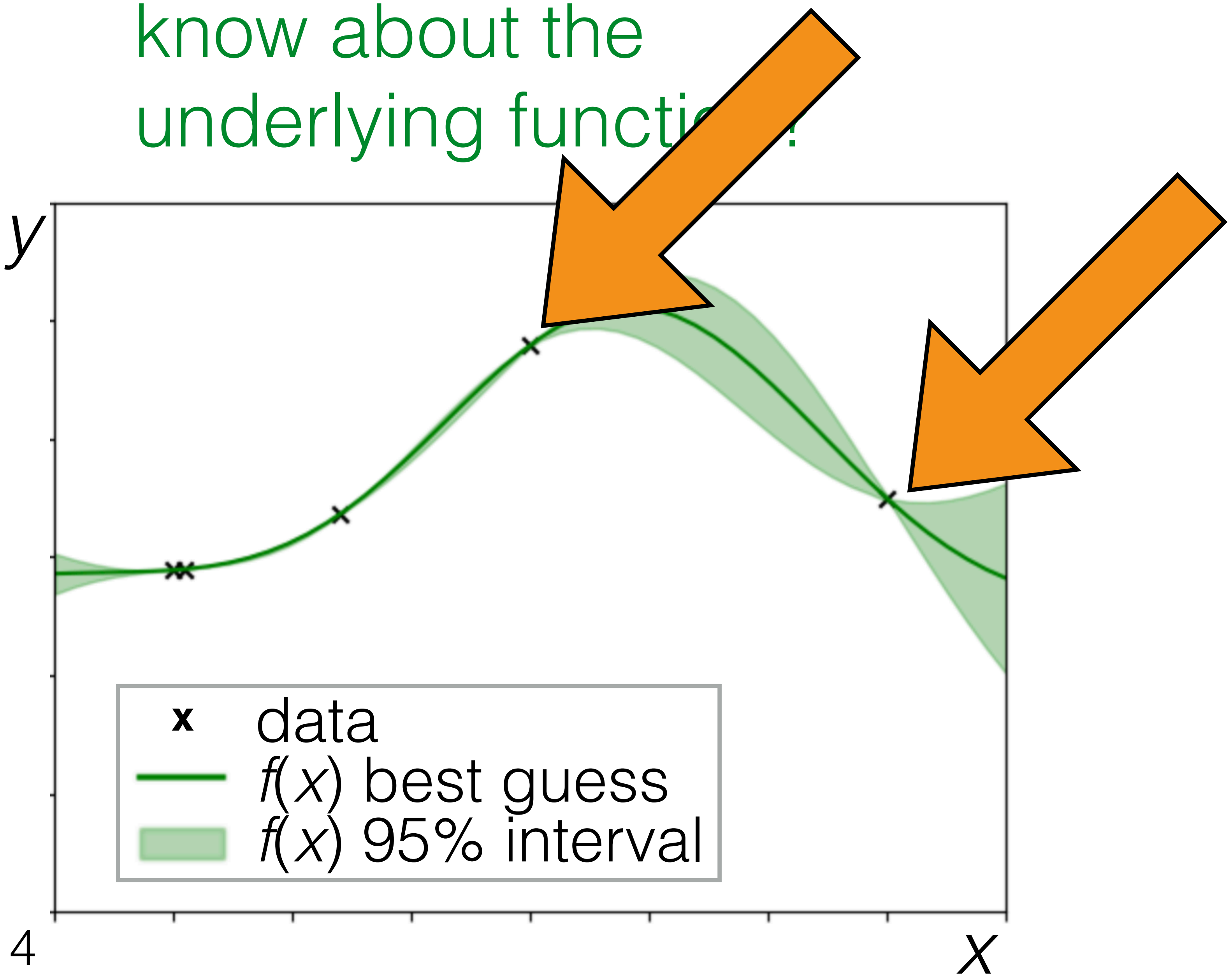

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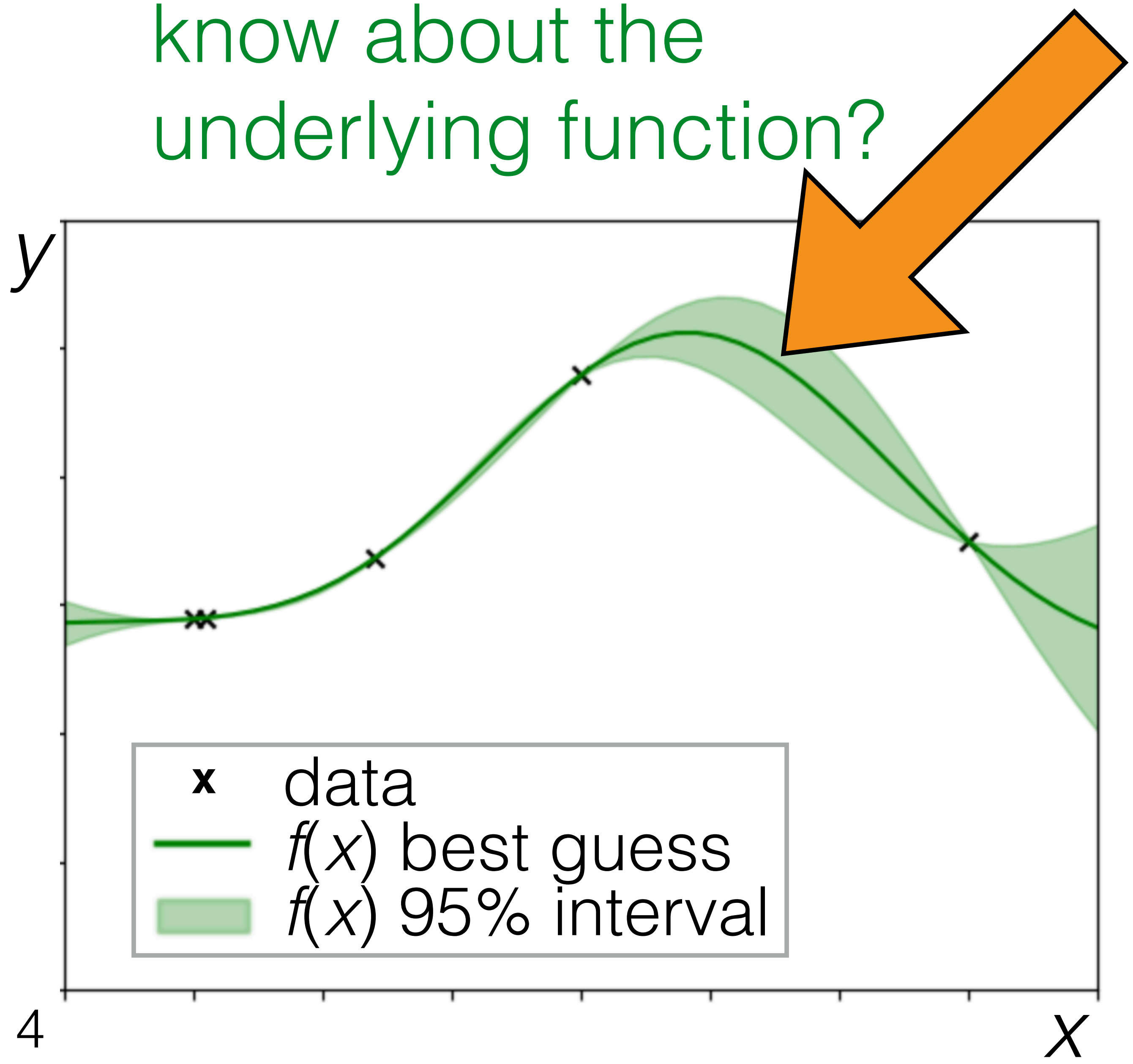

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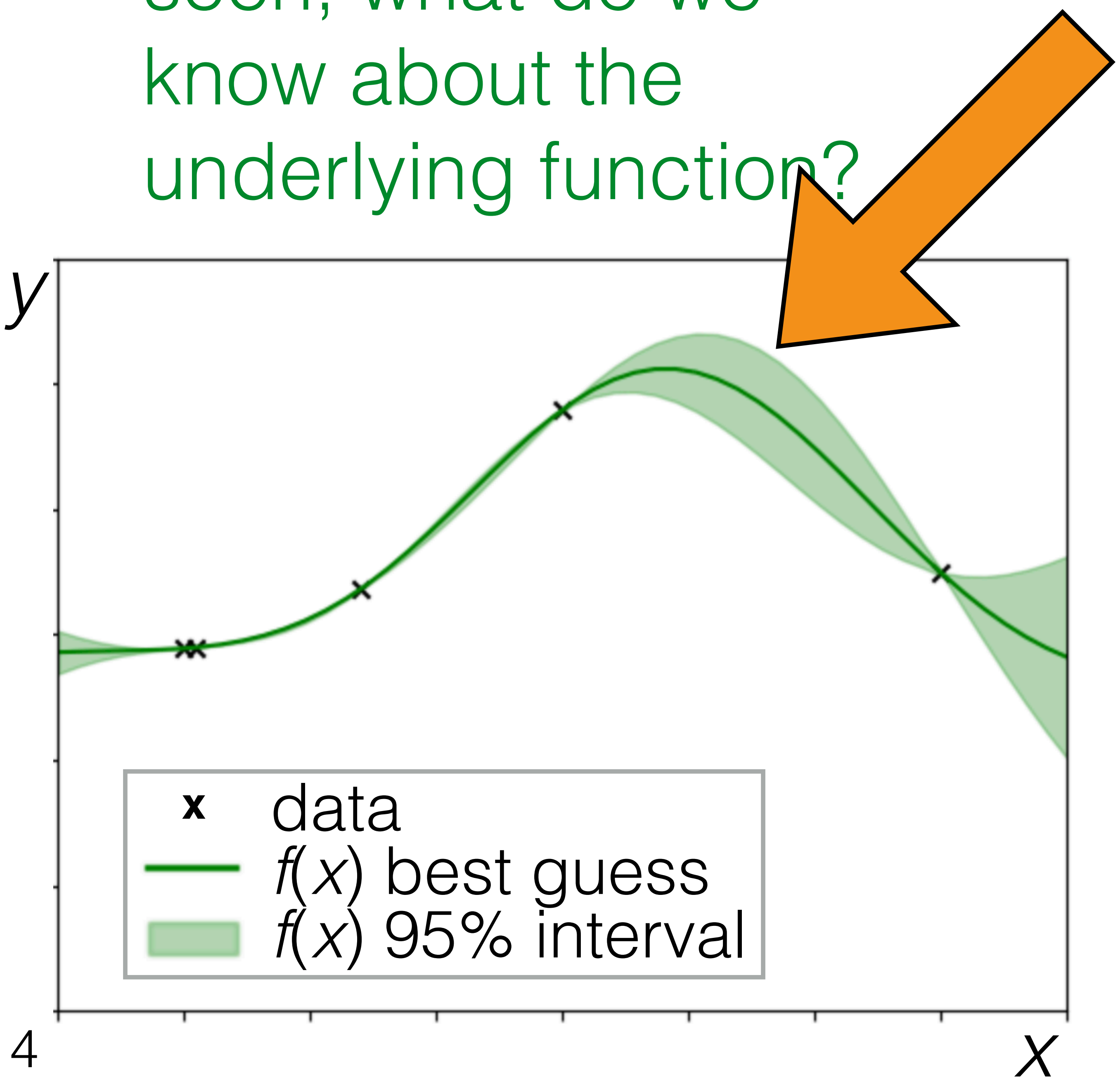

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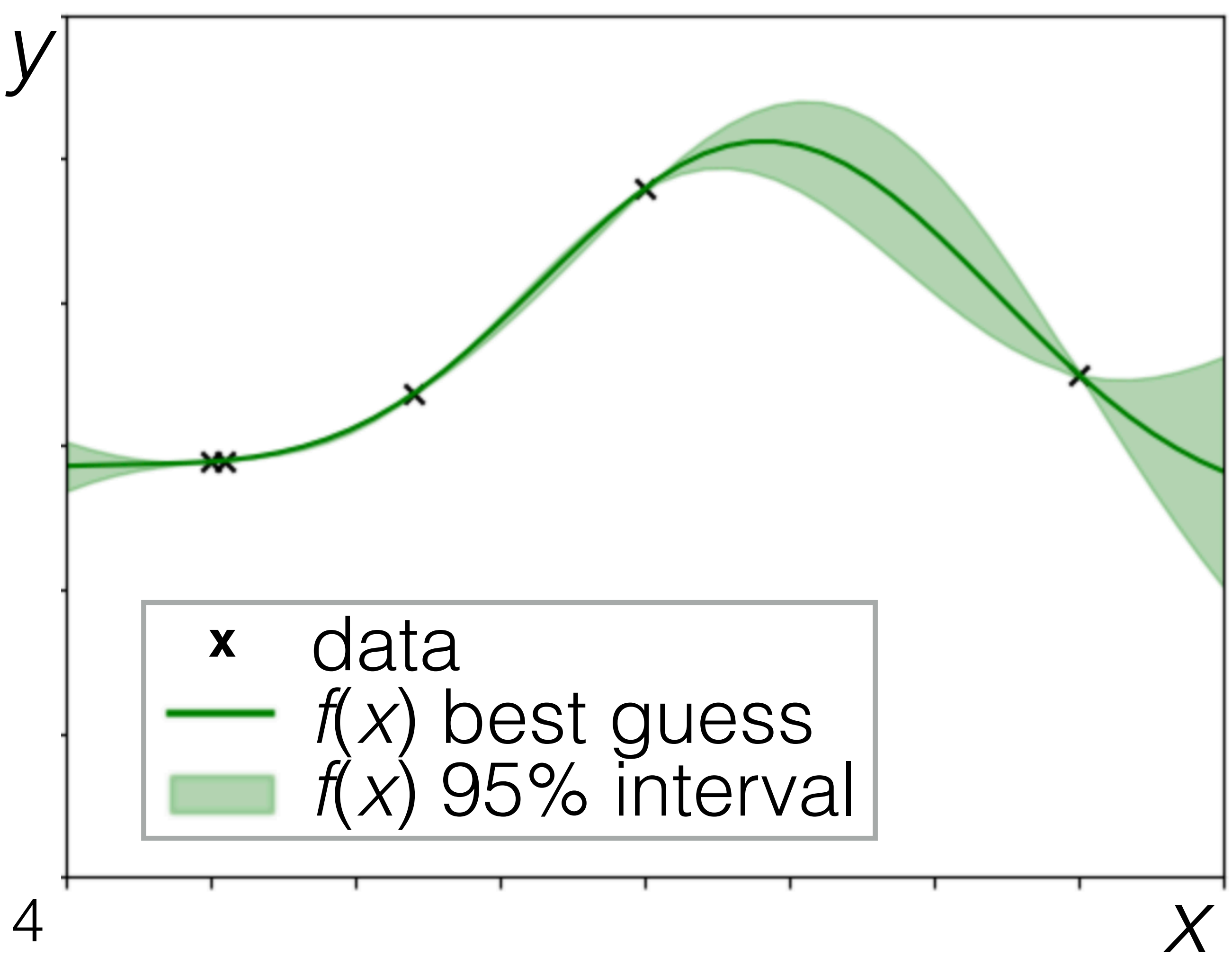

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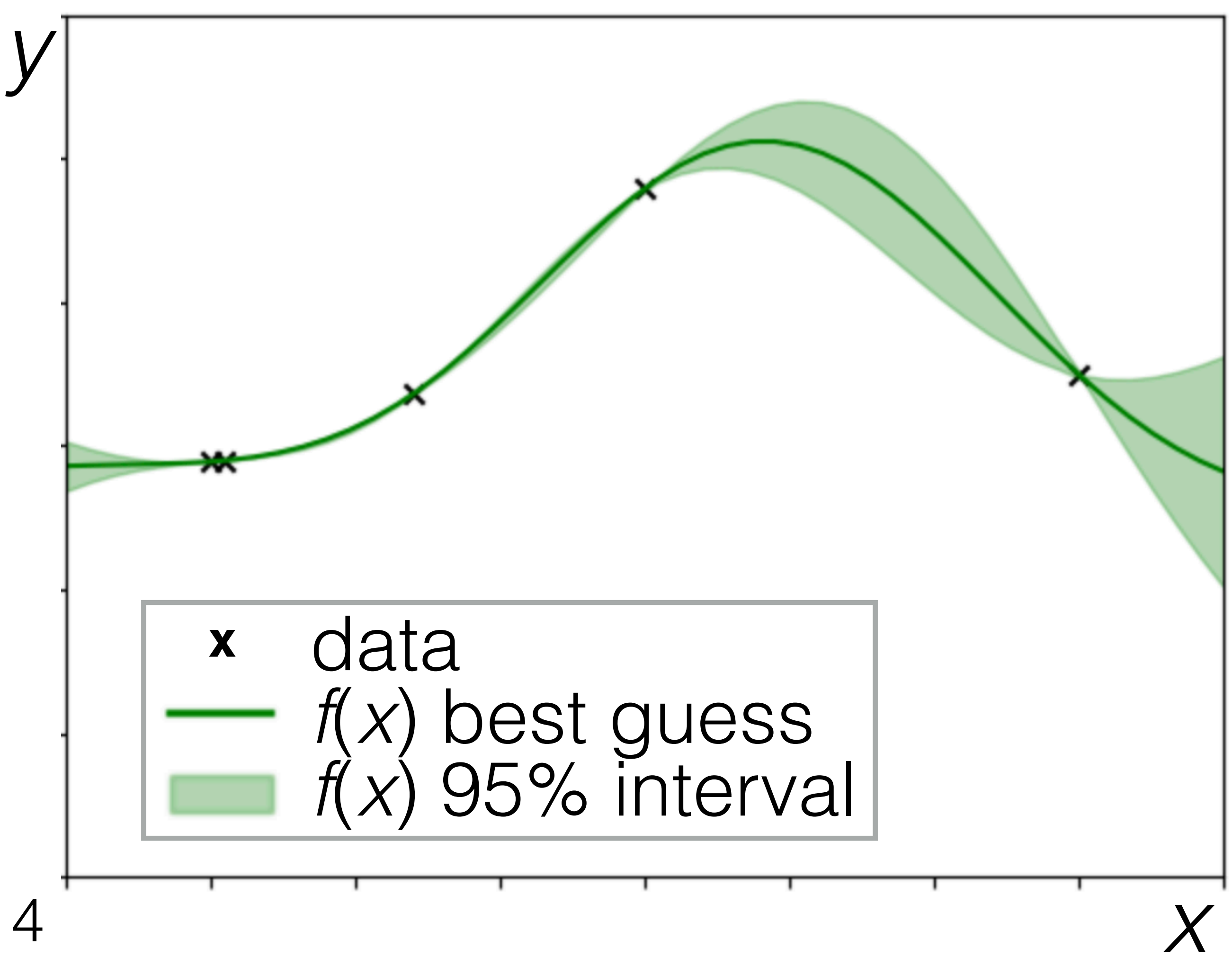

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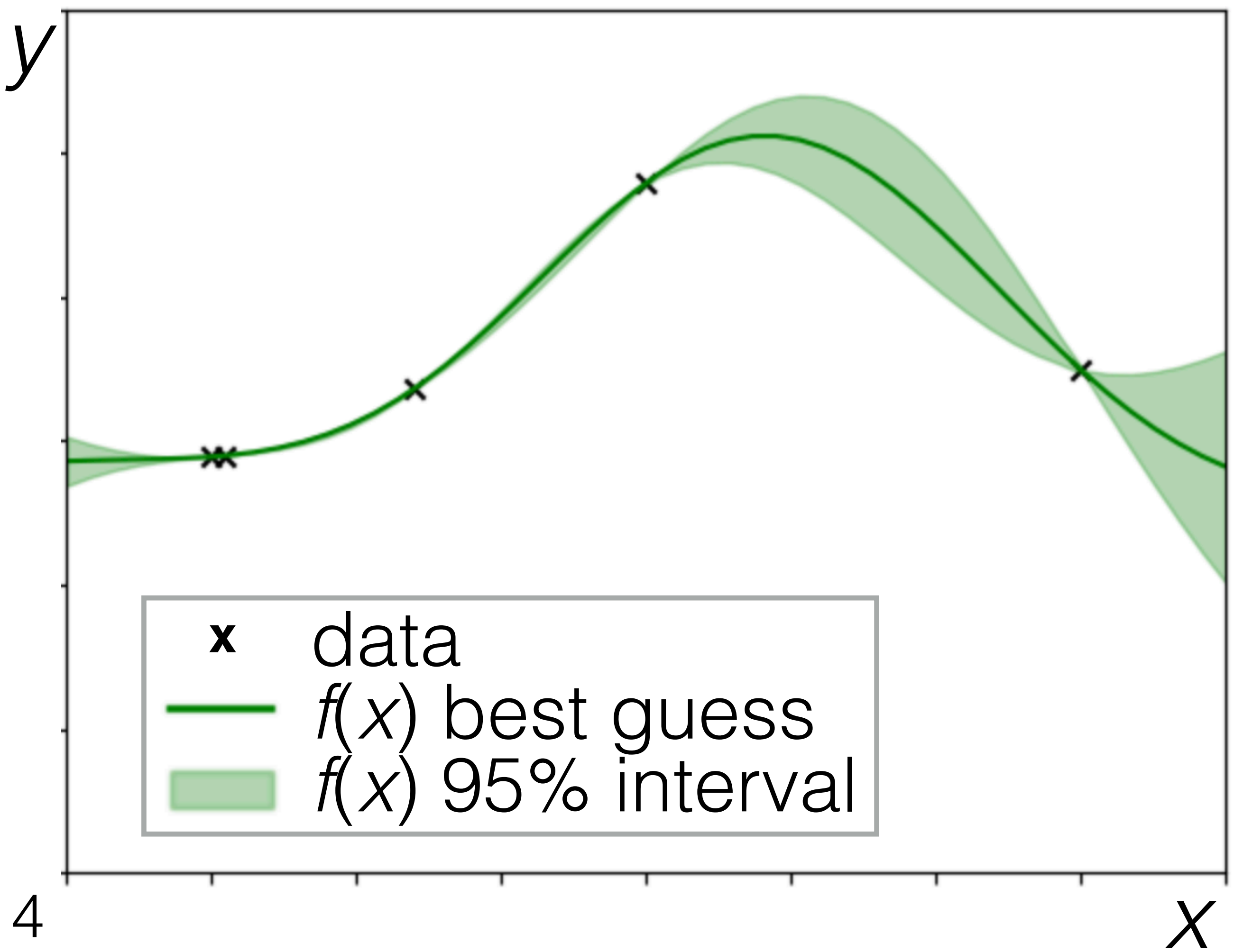

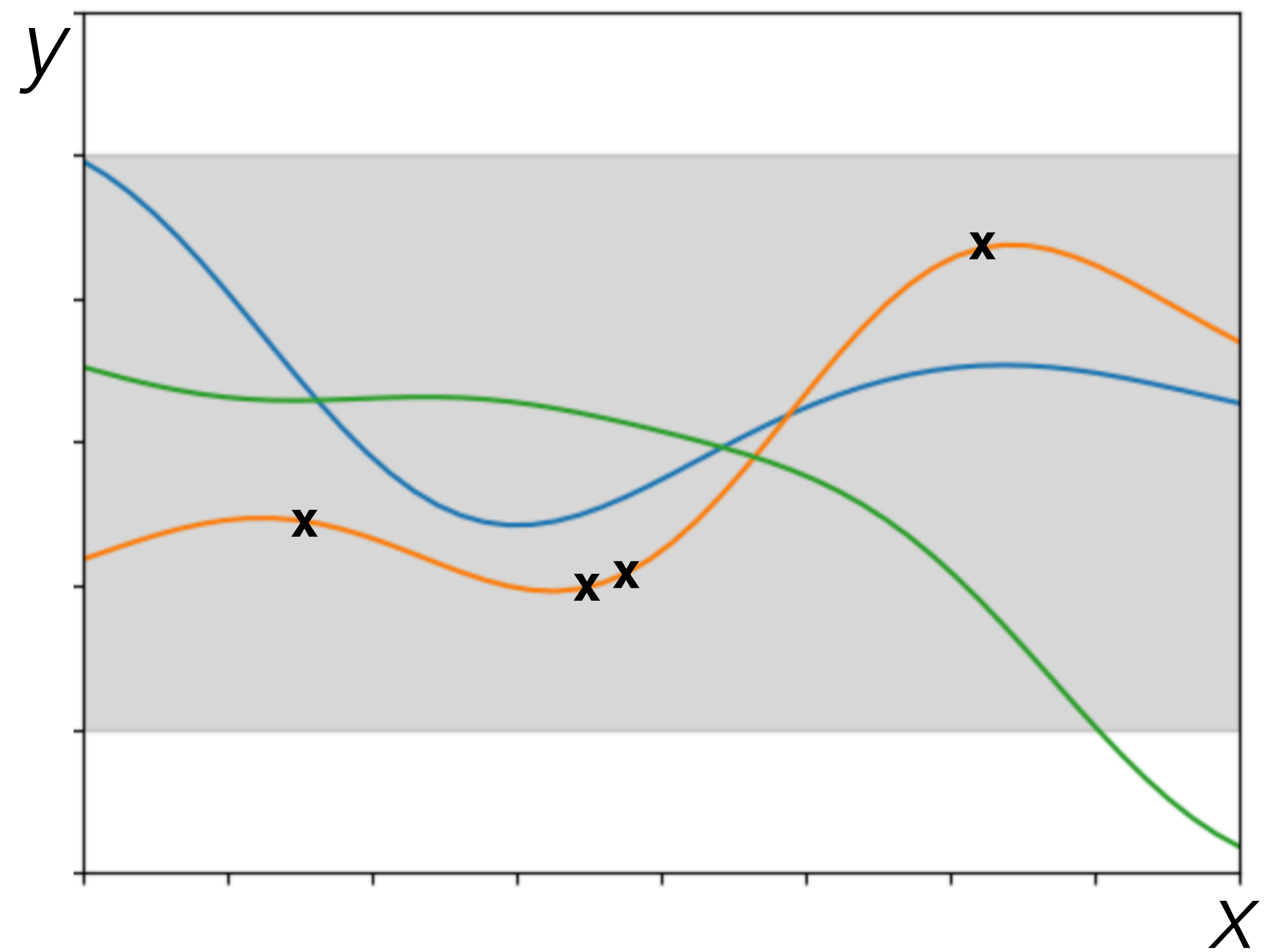

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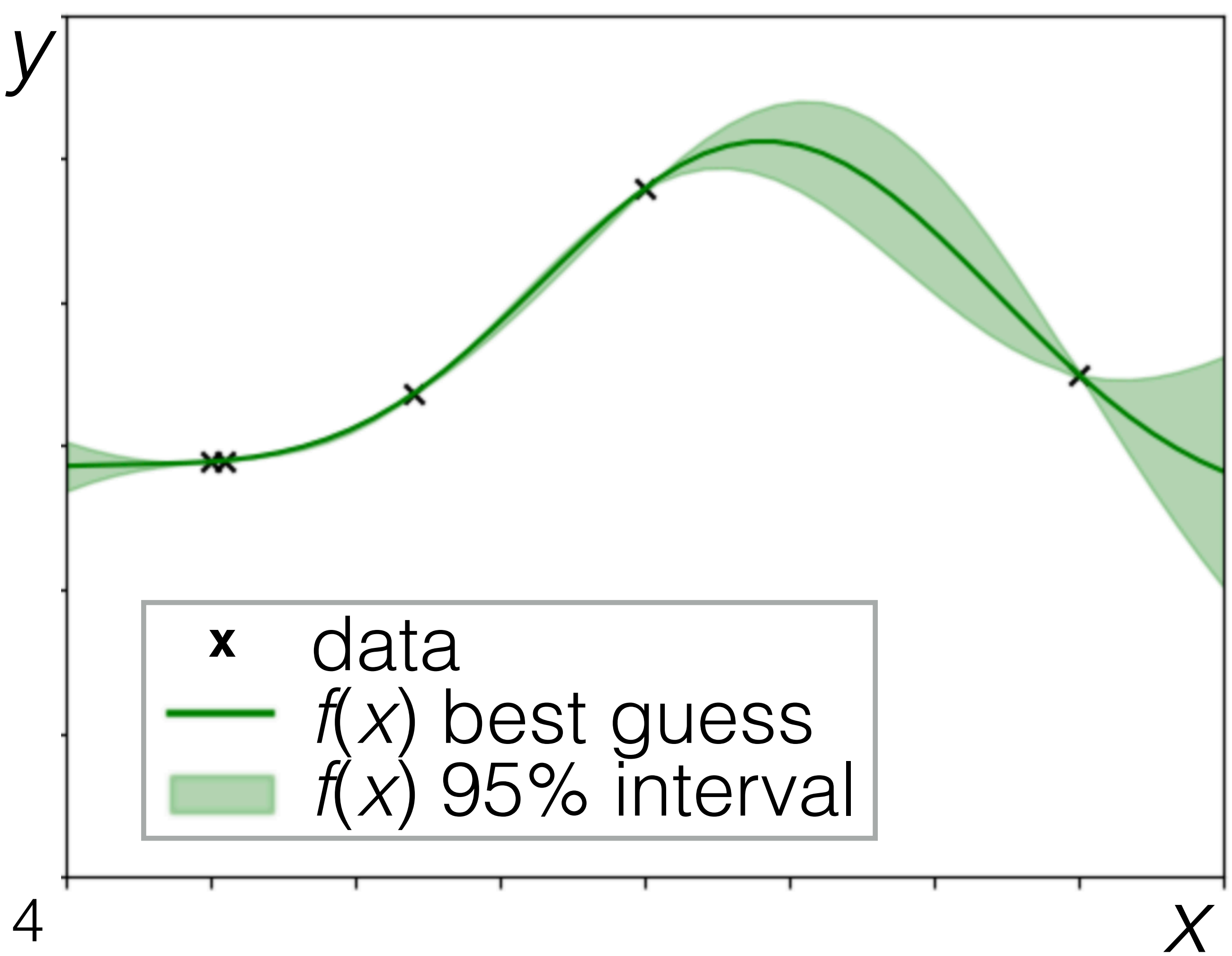

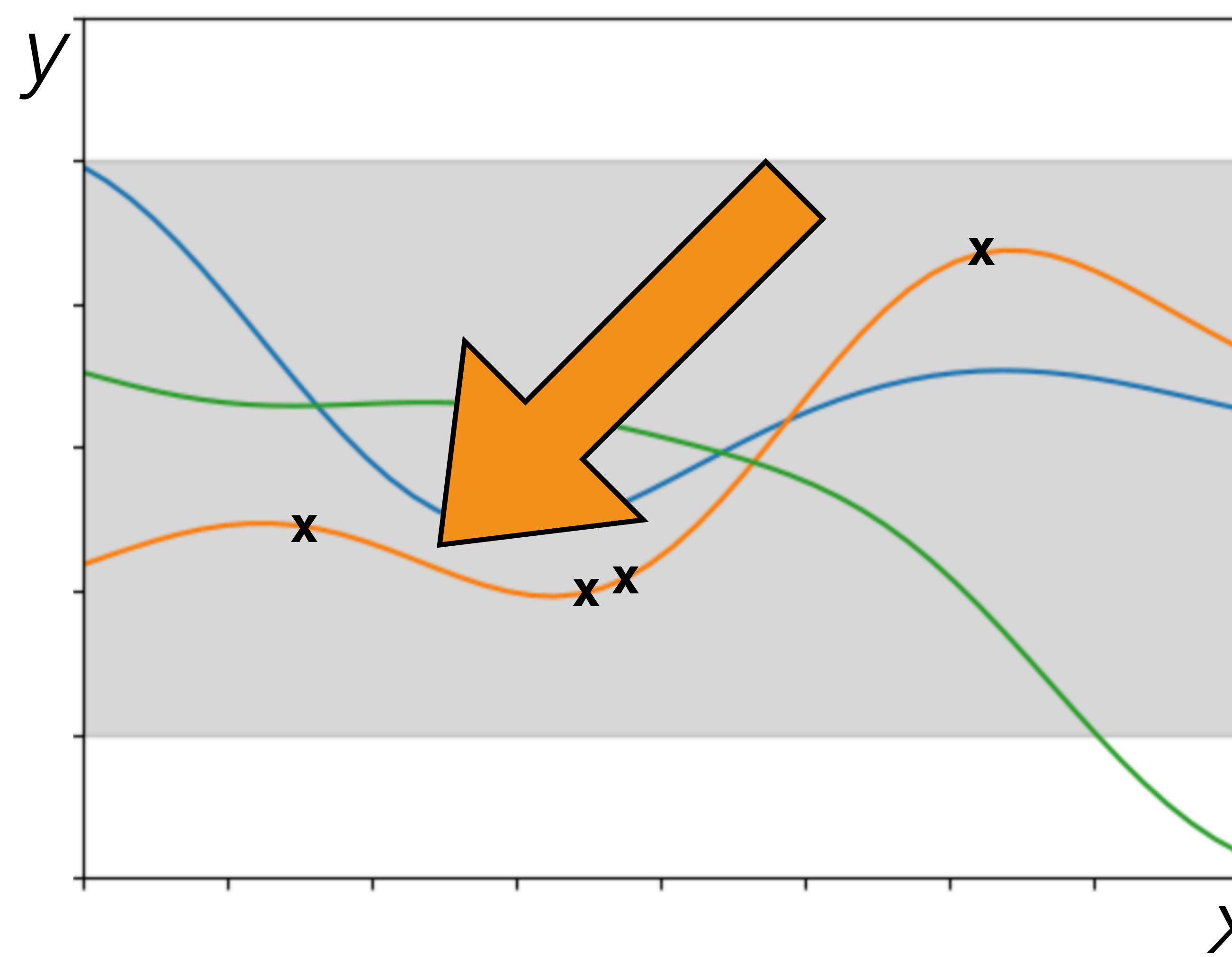

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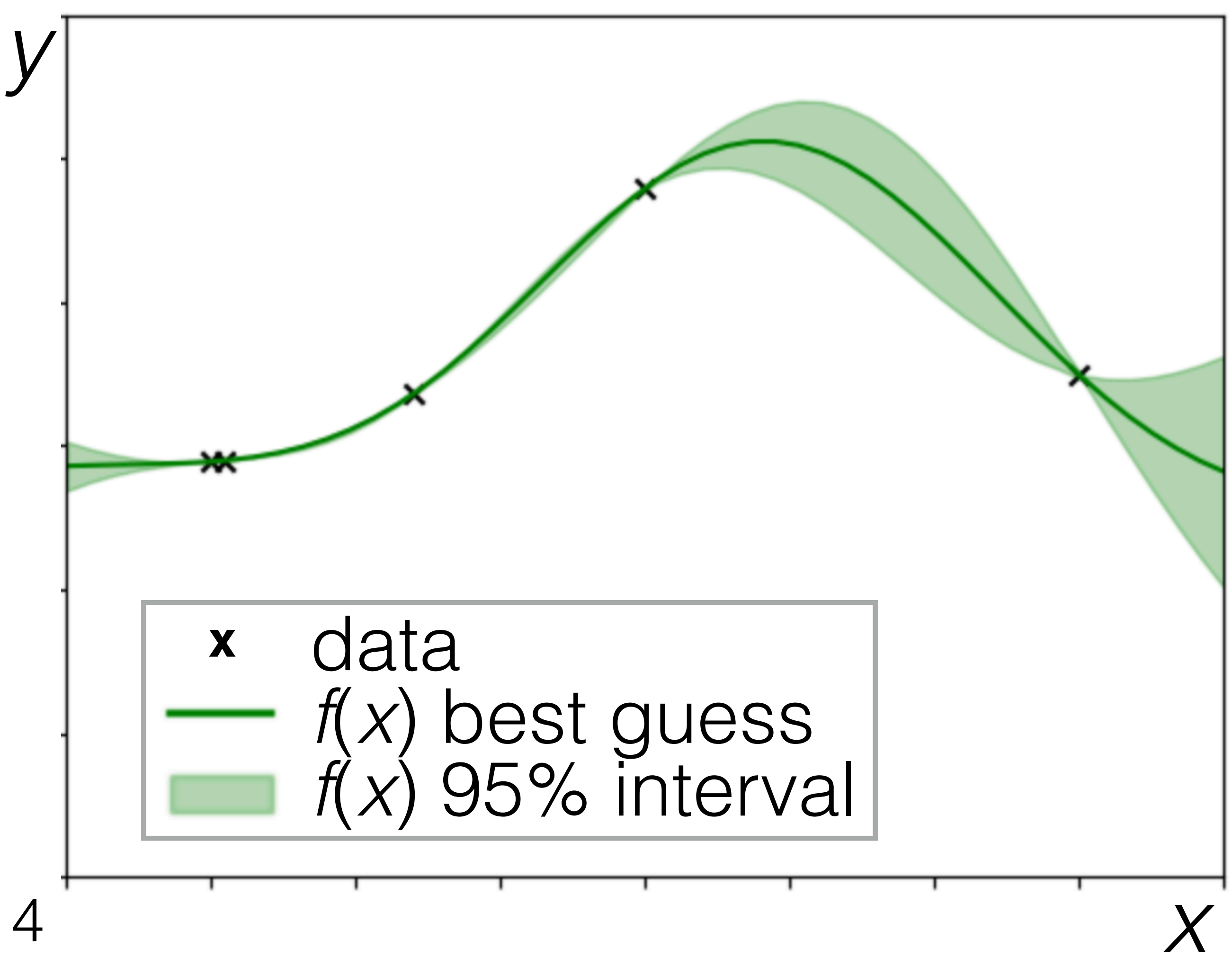

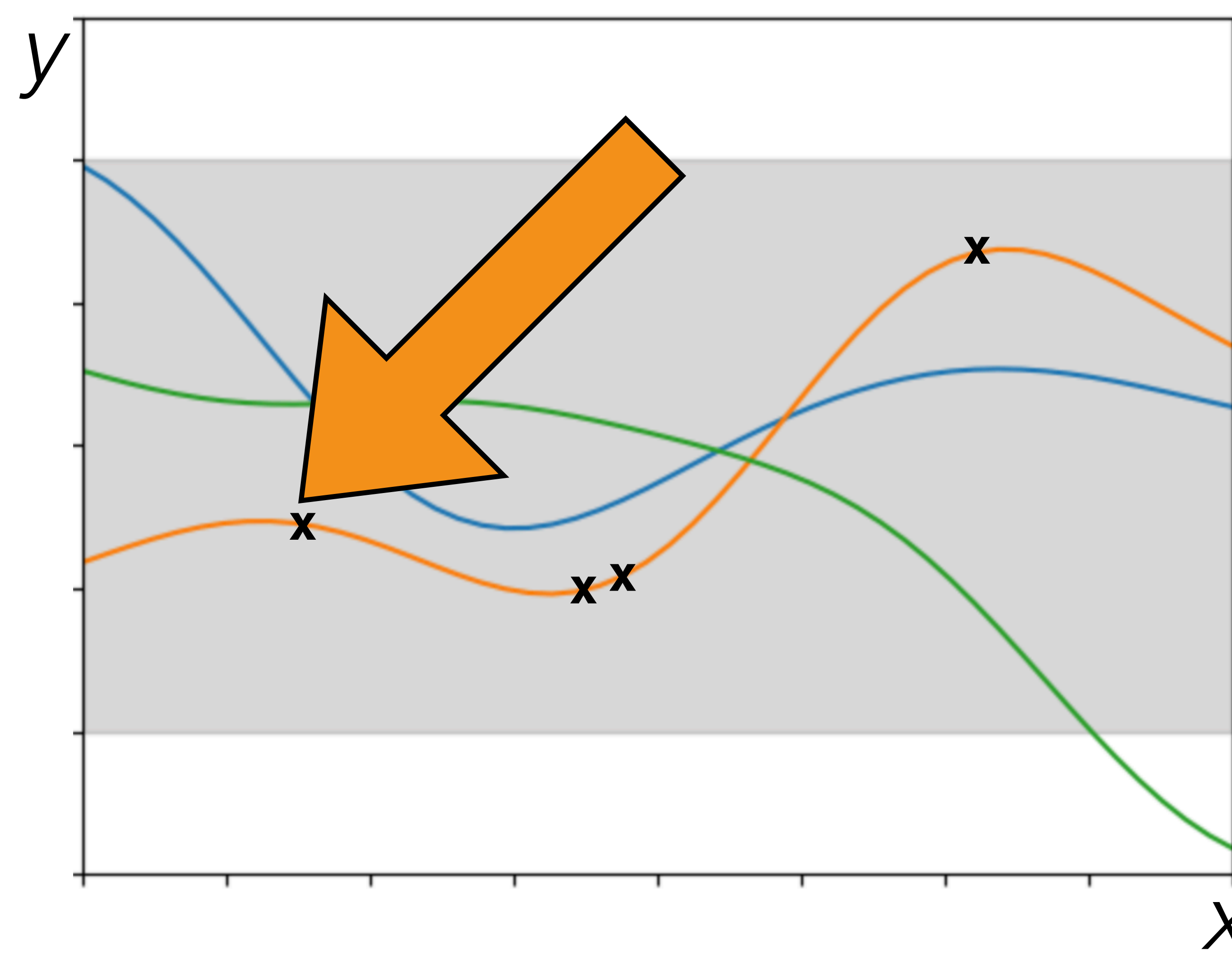

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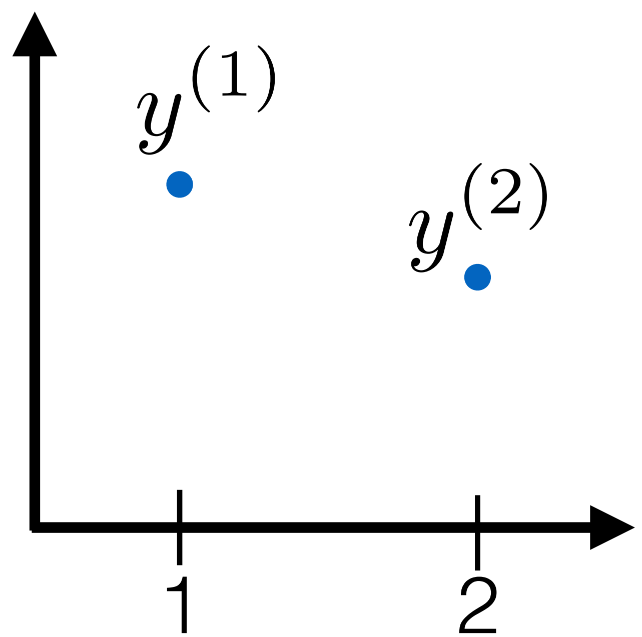

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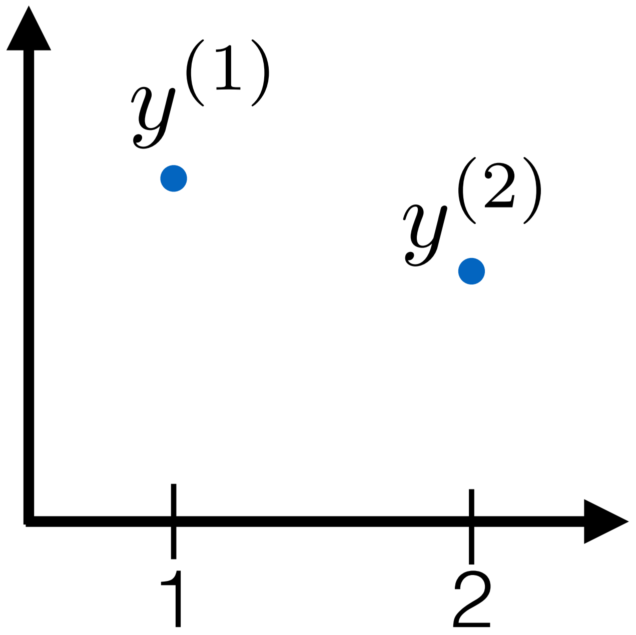

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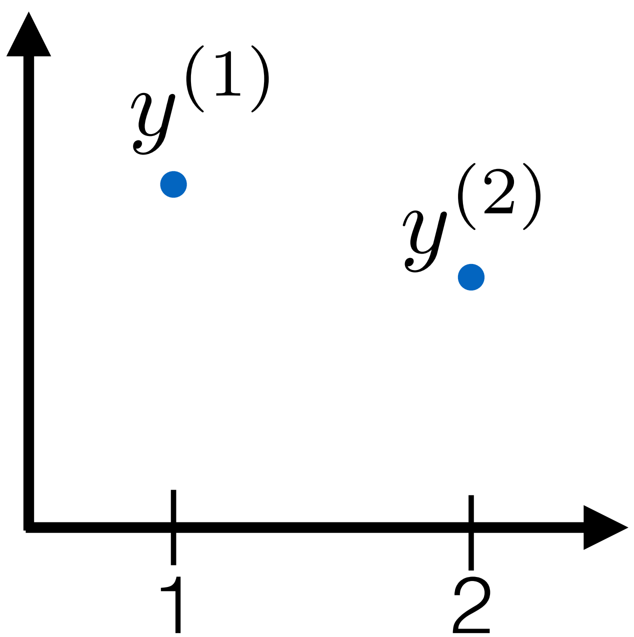

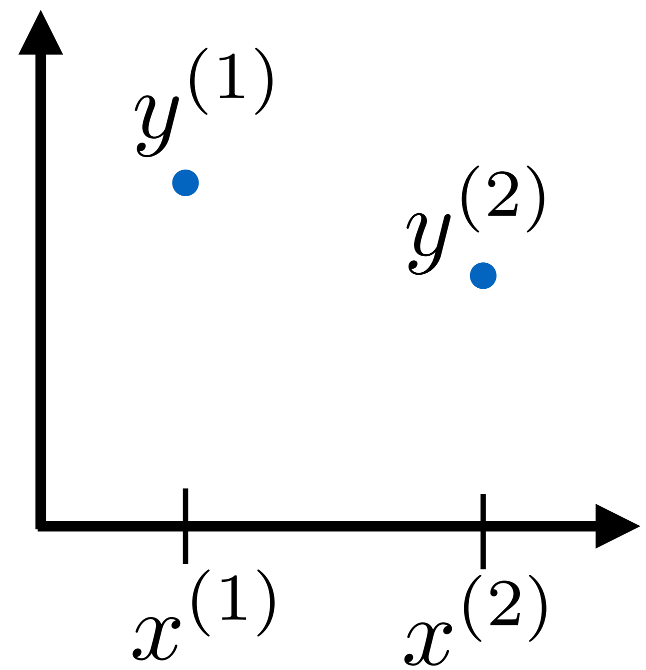

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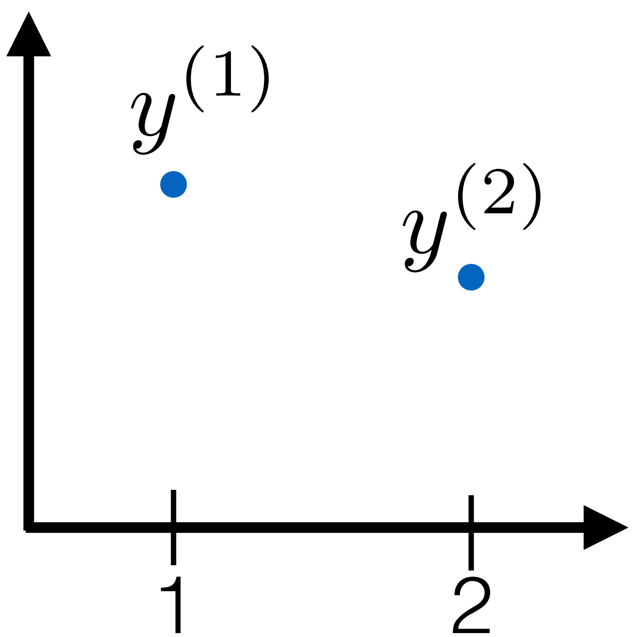

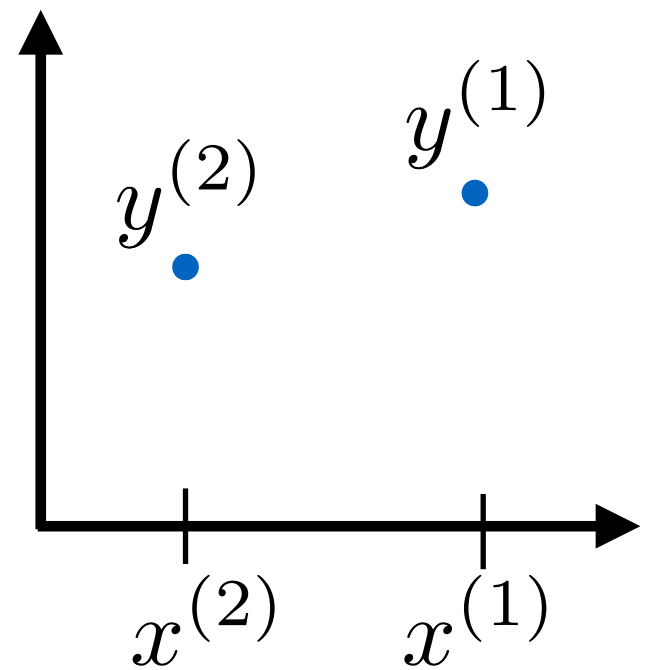

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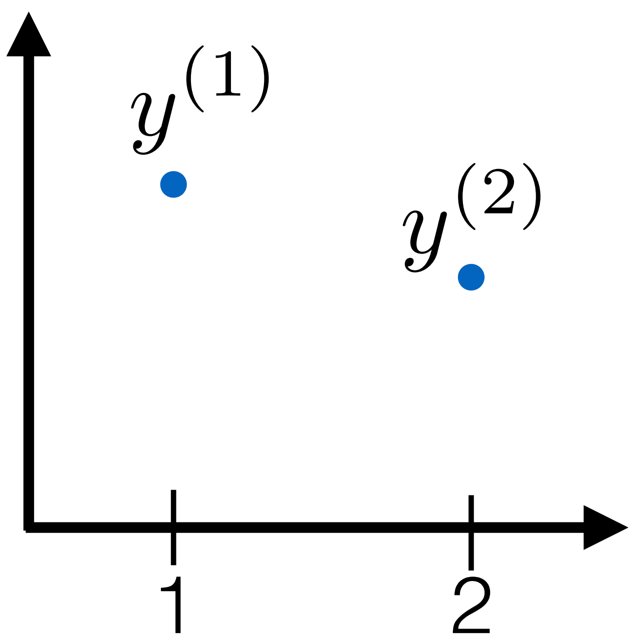

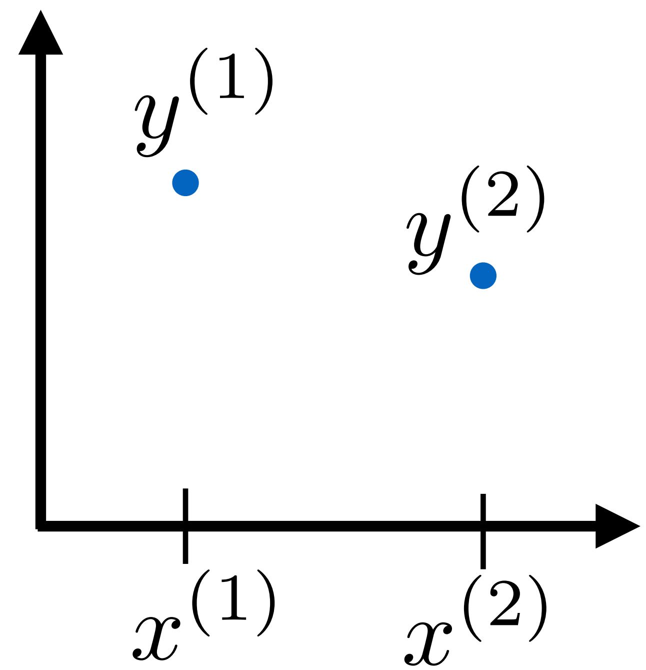

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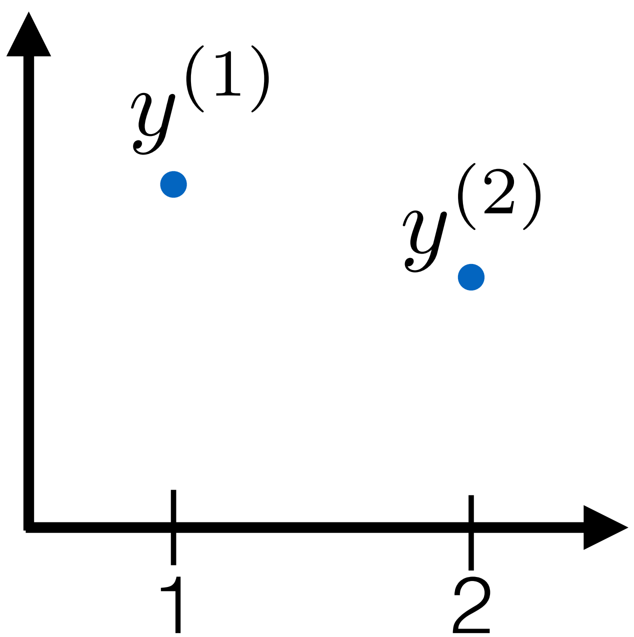

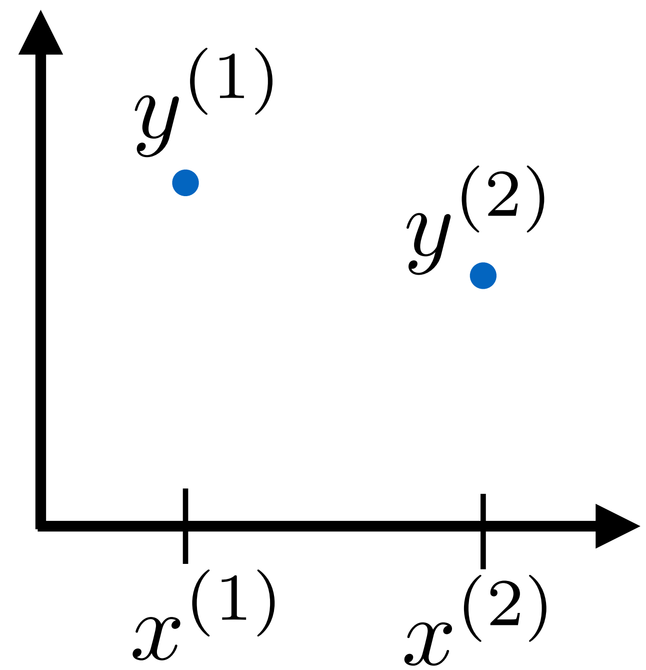

- What if we let the correlation depend on the x's?

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- M=2, bivariate Gaussian: $[y^{(1)}, y^{(2)}]^\top \sim \mathcal{N}(\mu, K)$
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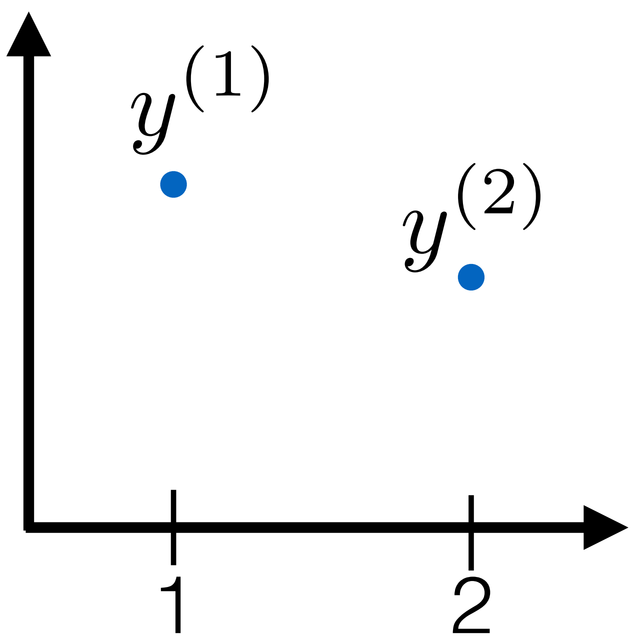

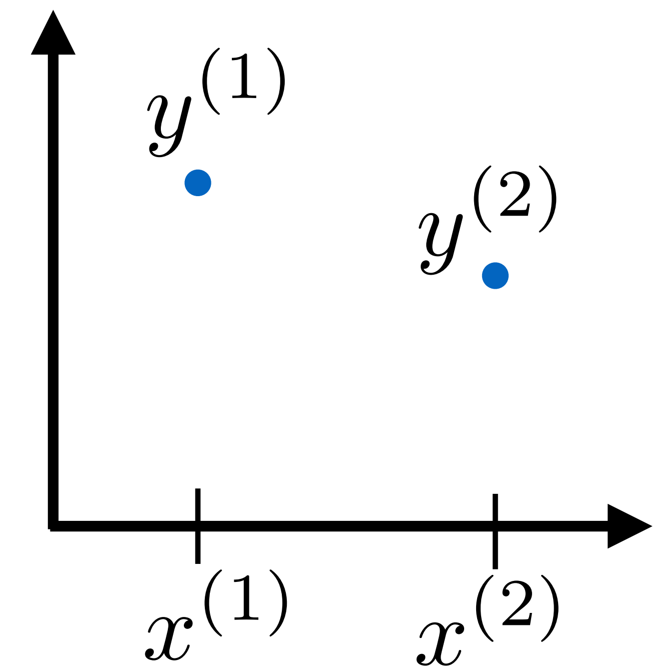

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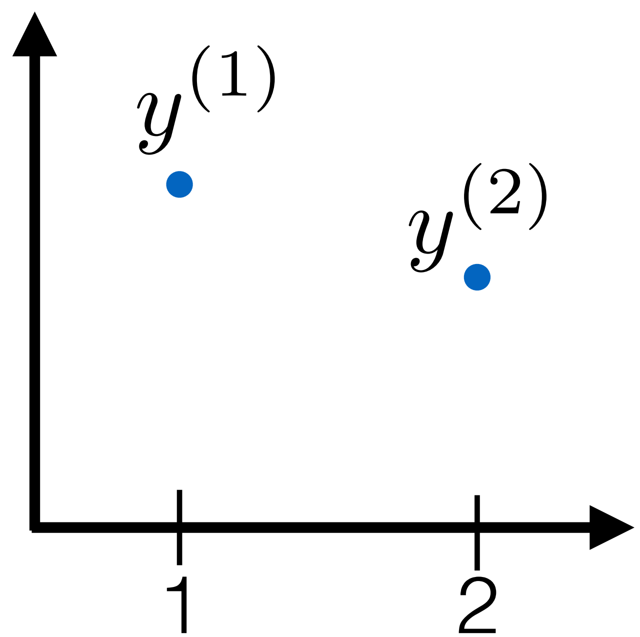

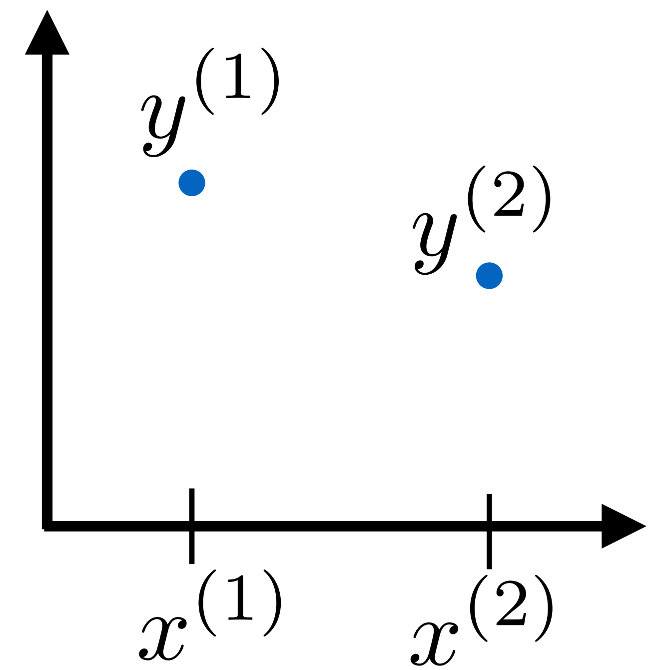

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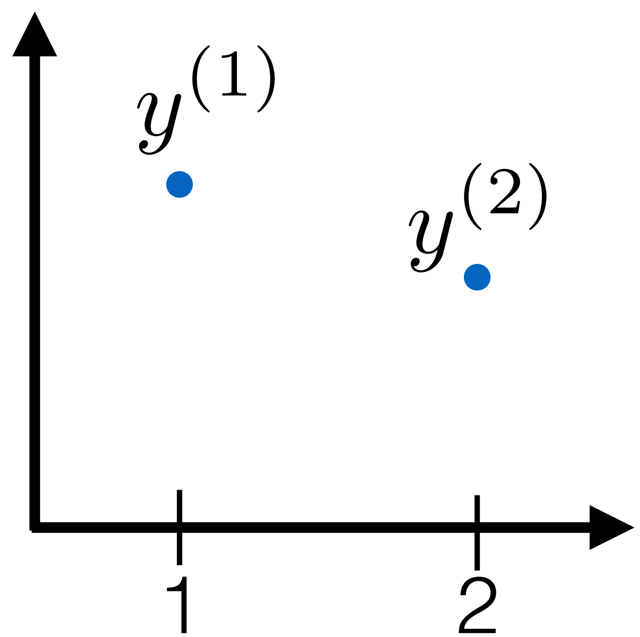

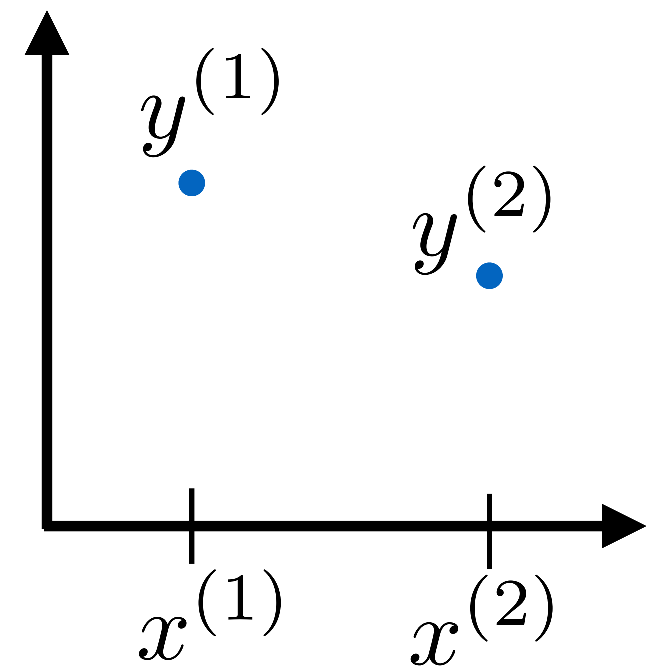

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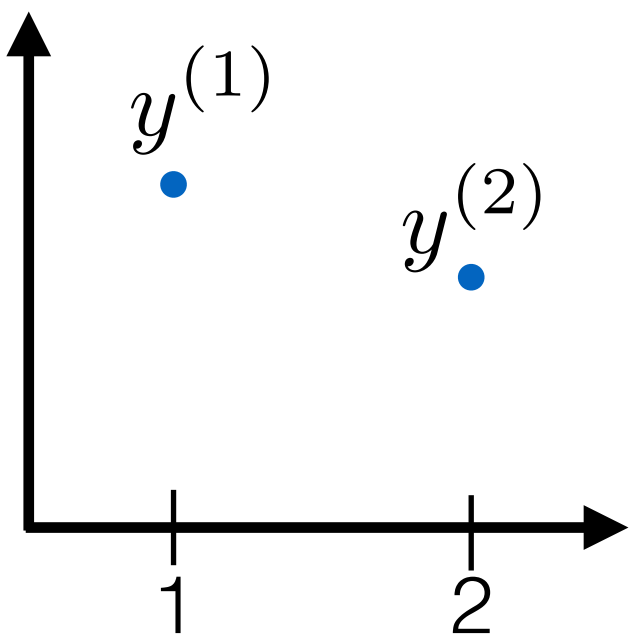

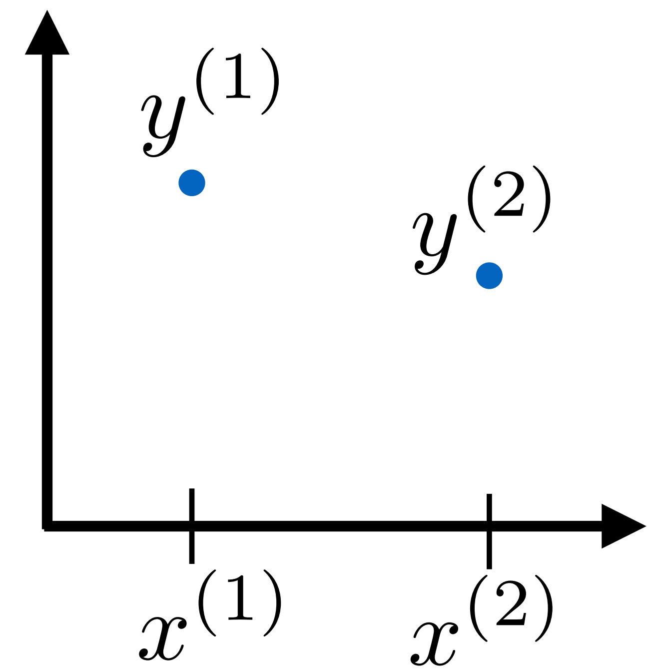

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[demo]

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We just drew random functions from a type of "Gaussian process"!

Gaussian processes

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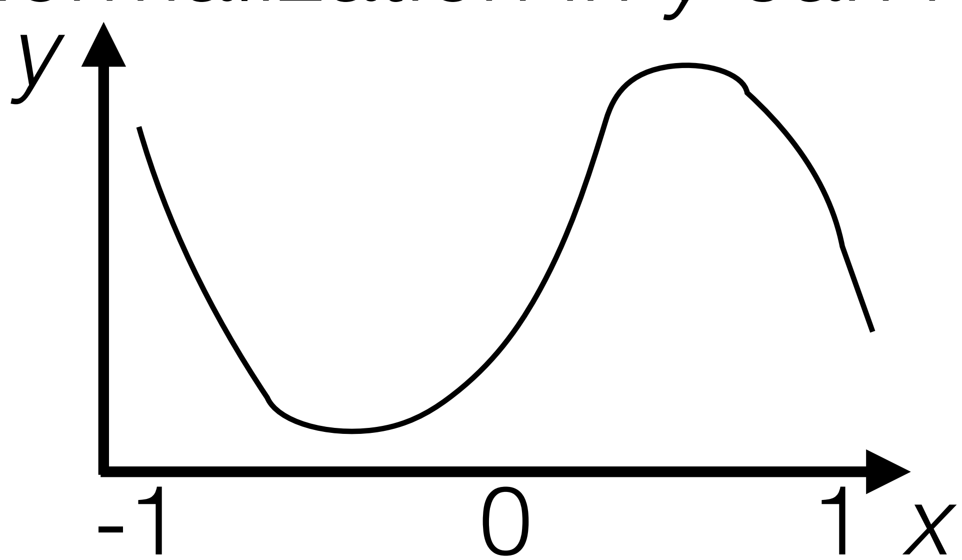

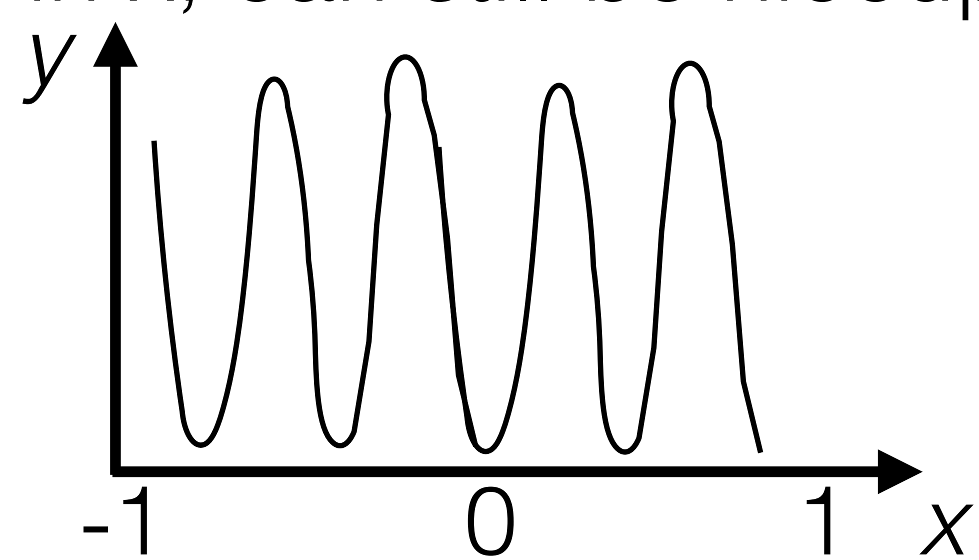

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Goal:

- Learn the mechanism behind standard GPs to identify benefits and pitfalls

Resources

http://www.tamarabroderick.com/tutorials.html

- Rasmussen and Williams 2006. *Gaussian Processes for Machine Learning*. https://gaussianprocess.org/gpml/
  - Chapters 1, 2, 4, 5
- Gramacy 2020. *Surrogates: Gaussian process modeling, design and optimization for the applied sciences*. https://bookdown.org/rbg/surrogates/
- Garnett 2023. *Bayesian Optimization*. https://bayesoptbook.com/
- Software options include:
  - scikit-learn, GPy, GPflow, GPyTorch
- My setup for this tutorial: pip install X
  - X = jupyterlab, notebook, numpy, matplotlib, scikit-learn

References (1/1)

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