Mean Squared Errors of X-11 Trend Filters

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Approximating X-11 with model-based filters

Cleveland (1972)

Cleveland and Tiao (1976)

Burridge and Wallis (1984)

Chu (2000)

Planas and Depoutot (2002)

used airline model with canonical decomposition

Approximating model-based filters with X-11

Depoutot and Planas (1998):

For a range of airline models, select X-11 MAs to

$$min \sum_j (\omega_{X11,j} - \omega_{air,j})^2$$

for the nonseasonal (N_t) and trend (T_t) components.

Chu, Tiao, and Bell (2012) and Bell, Chu, and Tiao (2012):

For a range of airline models, select X-11 MAs to

$$min \; E[N_t - \omega_{X11}(B) y_t]^2$$

We did not consider choosing X-11 MAs to minimize the MSE of X-11 trend estimates.

Wright (2017) and Pang, Monsell, and Bell (in progress) used simulations to study this further

Models for trend estimation

1. For a nonseasonal series

$$y_t = T_t + I_t \qquad I_t \sim i.i.d.\ N(0, \sigma_I^2)$$
$$(1-B)^2 T_t = \eta(B) b_t \qquad b_t \sim i.i.d.\ N\left(0, \sigma_b^2\right)$$

2. For a monthly seasonal series

$$y_t = T_t + S_t + I_t$$
$$(1-B)^2 T_t = \eta(B) b_t \qquad b_t \sim i.i.d.\ N\left(0, \sigma_b^2\right)$$
$$U(B) S_t = \eta_S(B) c_t \quad U(B) = (1 + B + \cdots + B^{11}) \quad c_t \sim i.i.d.\ N(0, \sigma_c^2)$$

Typical trend models have $\eta(B) = 1 - \eta_1 B - \eta_2 B^2$. Sometimes $\eta_2 = 0$ or $\eta_1 = \eta_2 = 0$. Such models were used by Akaike (1980), Hillmer and Tiao (1982), Gersch and Kitagawa (1983), Harvey (1989), Durbin and Koopman (2001), Hodrick and Prescott (1997), and Whittaker (1923).

Canonical decomposition of the airline model

Consider a time series y_t that follows the airline model:

$$(1-B)(1-B^{12})\, y_t = (1-\theta_1 B)(1-\theta_{12} B^{12} B) a_t \qquad a_t \sim i.i.d.\ N_t(0, \sigma_a^2)$$

Hillmer and Tiao (1982) showed that this model can be decomposed to give models for the components in $y_t = T_t + S_t + I_t$ of the form given on the previous slide, with $\eta_S(B)$ of degree 11, and σ_I^2 taking the maximum value possible that keeps the decomposition consistent with the original model. They call this the *canonical decomposition*.

The canonical trend model is of the form

$$(1-B)^d T_t = (1+B)(1-\eta B) b_t$$

where η is very close to $\theta_{12}^{1/12}$ except for values of $\theta_1 > 0.8$ and $\theta_{12} \leq 0.2$. Thus, apart from these exceptions, η depends on θ_{12} but little on θ_1. It exceeds 0.83 for all $\theta_{12} \geq 0.1$.

Note: The trend and irregular variances, σ_b^2 and σ_I^2, and their ratios, vary with θ_1 and θ_{12}.

Computing the mean squared error of a trend estimate

Consider a seasonal time series

$$y_t = T_t + S_t + I_t$$

whose components follow models of the general forms given earlier, and a linear trend estimator

$$\hat{T}_t = \omega_T(B) y_t = \sum_j \omega_j y_{t-j} \,.$$

The error in $\hat{T}_t$ is:

$$\begin{aligned} T_t - \hat{T}_t &= T_t - \omega_T(B)[T_t + S_t + I_t] \\ &= [1 - \omega_T(B)]\, T_t - \omega_T(B) S_t - \omega_T(B) I_t \\ &= \left\{\frac{1-\omega_T(B)}{(1-B)^2}\right\} (1-B)^2 T_t - \left\{\frac{\omega_T(B)}{U(B)}\right\} U(B) S_t - \omega_T(B) I_t \end{aligned}$$

This error is stationary so that the MSE of $\hat{T}_t$ can be computed *if certain conditions are satisfied*.

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Computing the mean squared error of a trend estimate

The error in $\hat{T}_t$ is:

$$T_t - \hat{T}_t = \left\{\frac{1 - \omega_T(B)}{(1 - B)^2}\right\}(1 - B)^2 T_t - \left\{\frac{\omega_T(B)}{U(B)}\right\} U(B) S_t - \omega_T(B) I_t$$

This error is stationary so we can compute the MSE of $\hat{T}_t$ if

- $1 - \omega_T(B)$ contains $(1 - B)^2$
  - True for model-based filters from models with $(1 - B)^2$
  - True for X-11 symmetric filters
  - True for X-11 asymmetric filters obtained using the symmetric filters with **full forecast extension**.
- $\omega_T(B)$ contains $U(B)$ – always holds in practice

Bell, Chu, and Tiao (2012) show how to compute the MSE from a different expression for the error.

Determining which Henderson MAs minimize the MSE of trend estimates

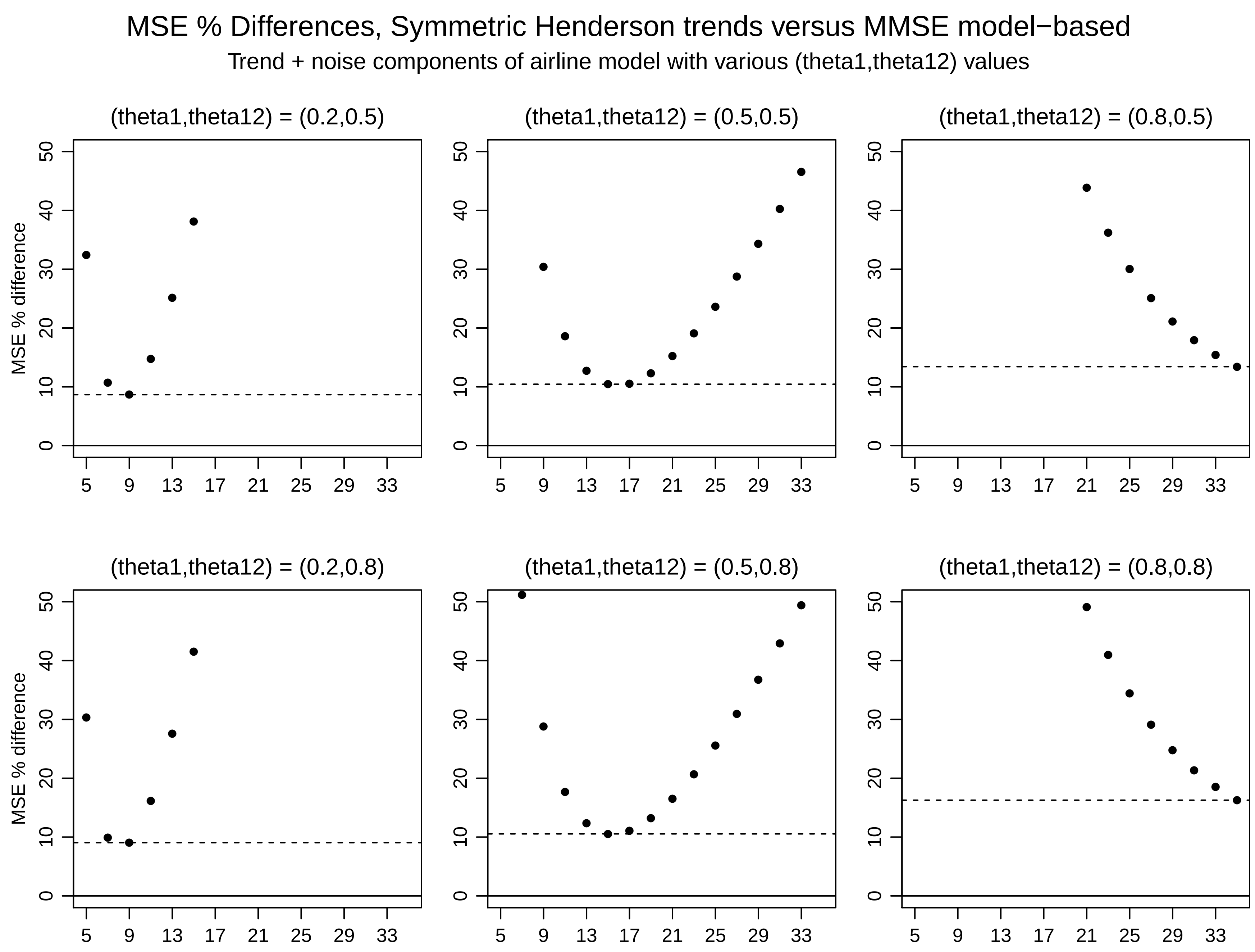
Two approaches were used:

1. Seasonal model: $y_t = T_t + S_t + I_t$, from canonical decomposition of the airline model.
2. Nonseasonal model: $y_t = T_t + I_t$, with T_t and I_t following the trend and irregular models obtained from canonical decomposition of the airline model.

This was done for airline models with parameters θ_1 and θ_{12} both ranging over $.1, \ldots, .9$, and with $\sigma_a^2 = 1$.

Results focus on MSE percentage differences from the optimal model-based estimator:

$$MSE\ \%\ diff = 100 \times \left(\frac{MSE\ X11}{MSE\ model\ based} - 1 \right)$$


MSE % Differences, Symmetric Henderson trends versus MMSE model−based
Trend + noise components of airline model with various (theta1,theta12) values
(theta1,theta12) = (0.2,0.5)
(theta1,theta12) = (0.5,0.5)
(theta1,theta12) = (0.8,0.5)
(theta1,theta12) = (0.2,0.8)
(theta1,theta12) = (0.5,0.8)
(theta1,theta12) = (0.8,0.8)
MSE % difference
0
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30
40
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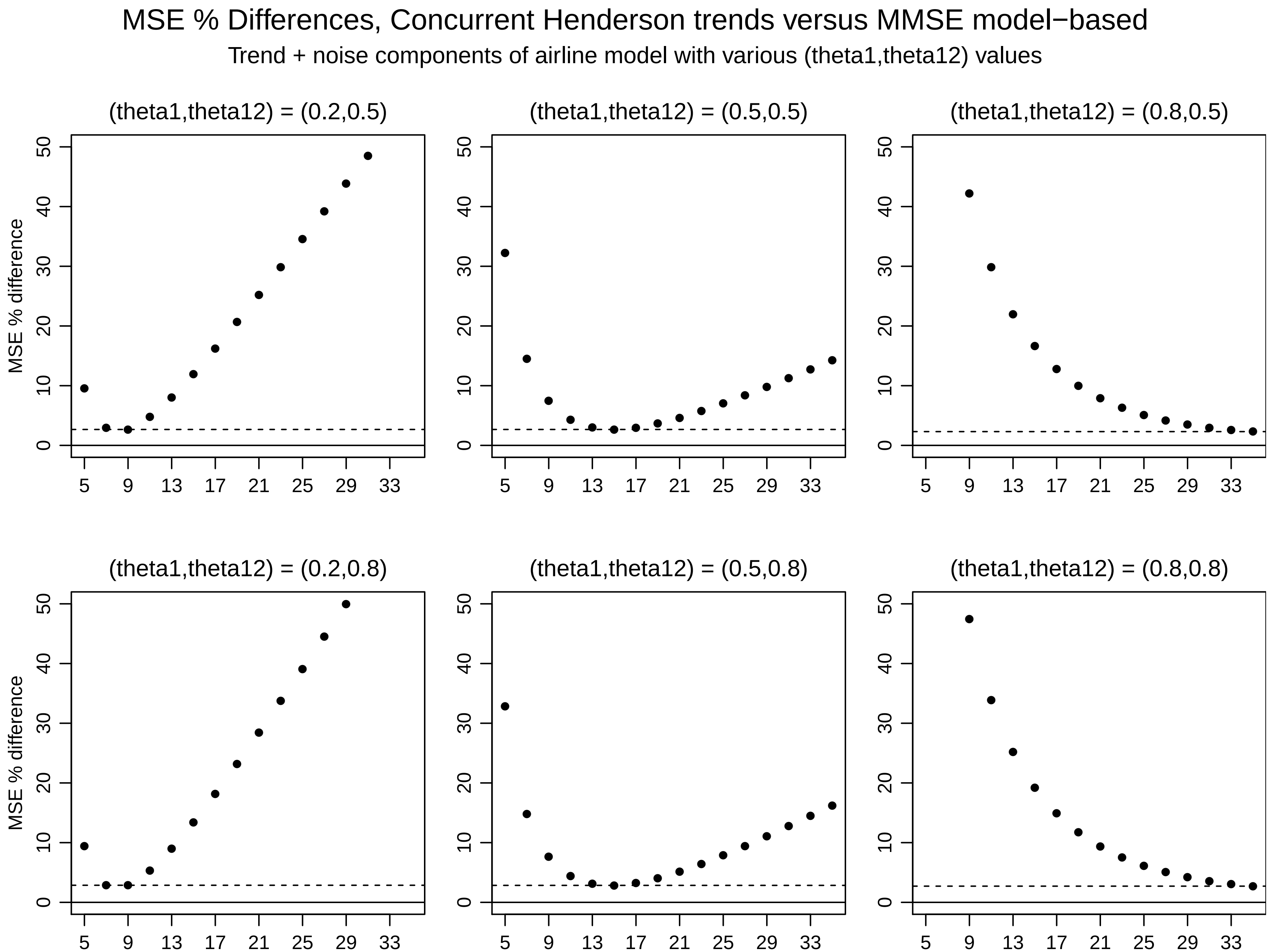
MSE % Differences, Concurrent Henderson trends versus MMSE model−based
Trend + noise components of airline model with various (theta1,theta12) values
(theta1,theta12) = (0.2,0.5)
(theta1,theta12) = (0.5,0.5)
(theta1,theta12) = (0.8,0.5)
(theta1,theta12) = (0.2,0.8)
(theta1,theta12) = (0.5,0.8)
(theta1,theta12) = (0.8,0.8)
MSE % difference
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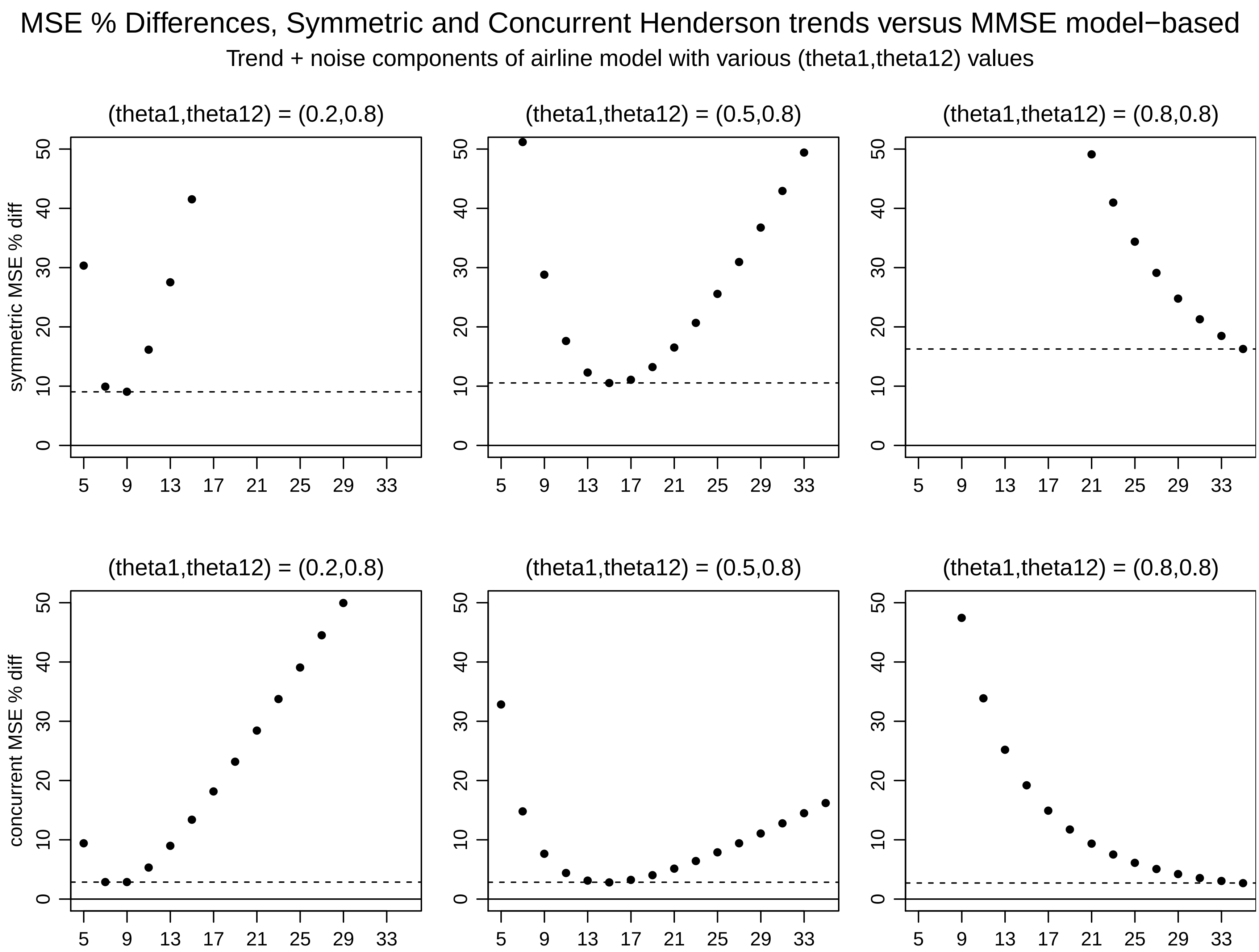
MSE % Differences, Symmetric and Concurrent Henderson trends versus MMSE model−based
Trend + noise components of airline model with various (theta1,theta12) values
(theta1,theta12) = (0.2,0.8)
(theta1,theta12) = (0.5,0.8)
(theta1,theta12) = (0.8,0.8)
symmetric MSE % diff
concurrent MSE % diff
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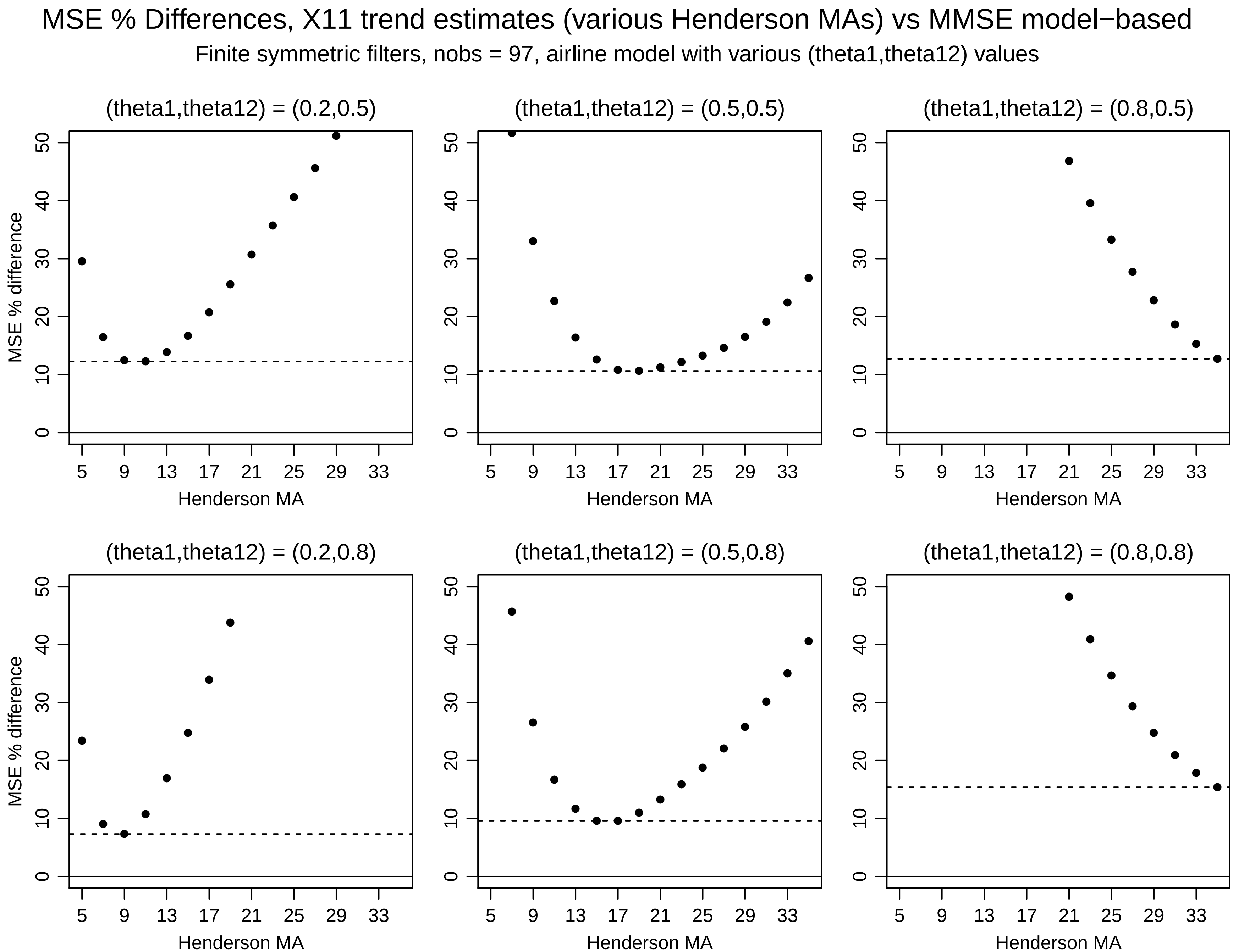
MSE % Differences, X11 trend estimates (various Henderson MAs) vs MMSE model−based
Finite symmetric filters, nobs = 97, airline model with various (theta1,theta12) values
(theta1,theta12) = (0.2,0.5)
(theta1,theta12) = (0.5,0.5)
(theta1,theta12) = (0.8,0.5)
(theta1,theta12) = (0.2,0.8)
(theta1,theta12) = (0.5,0.8)
(theta1,theta12) = (0.8,0.8)
MSE % difference
Henderson MA
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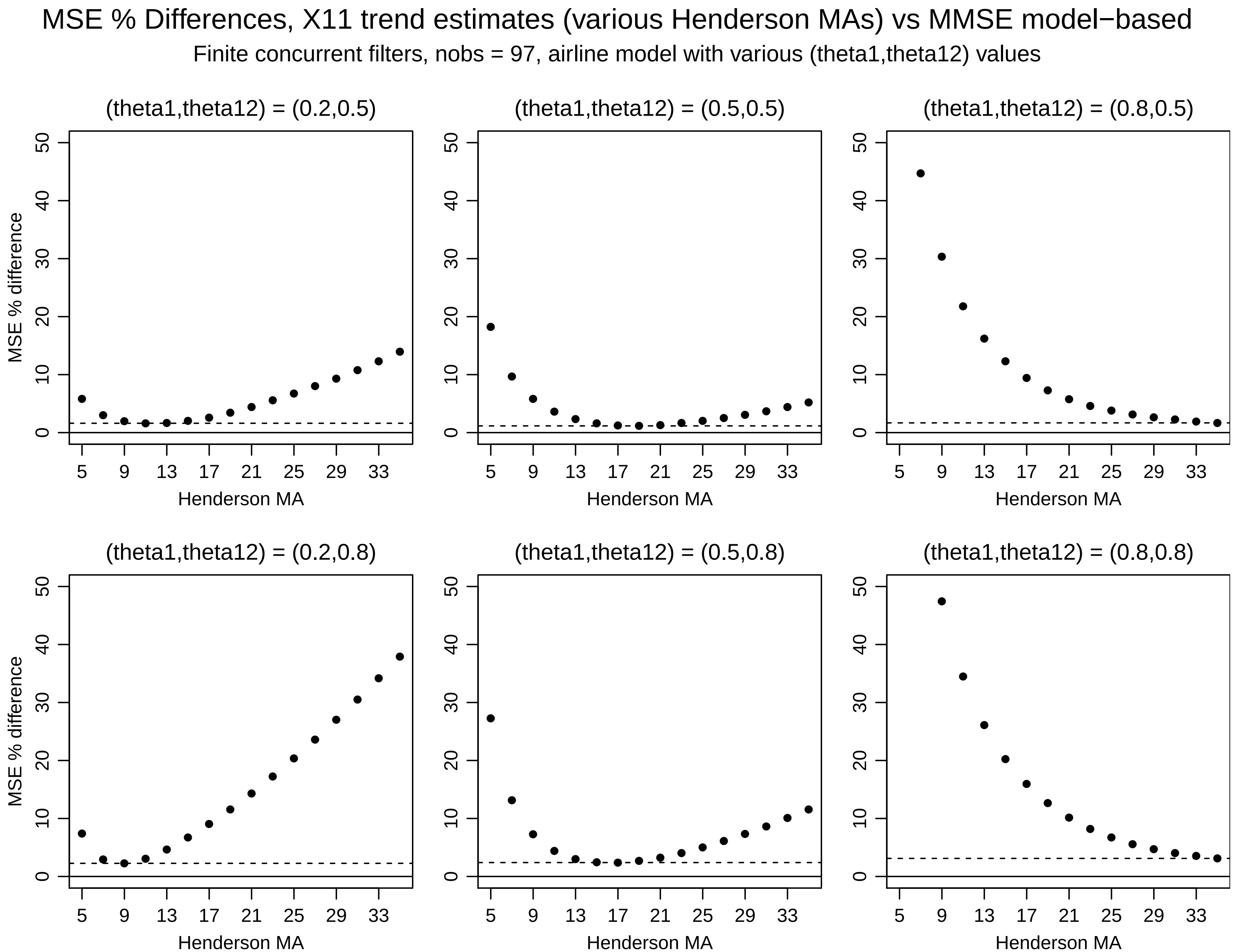

MSE % Differences, X11 trend estimates (various Henderson MAs) vs MMSE model−based
Finite concurrent filters, nobs = 97, airline model with various (theta1,theta12) values
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(theta1,theta12) = (0.5,0.5)
(theta1,theta12) = (0.8,0.5)
(theta1,theta12) = (0.2,0.8)
(theta1,theta12) = (0.5,0.8)
(theta1,theta12) = (0.8,0.8)
MSE % difference
Henderson MA
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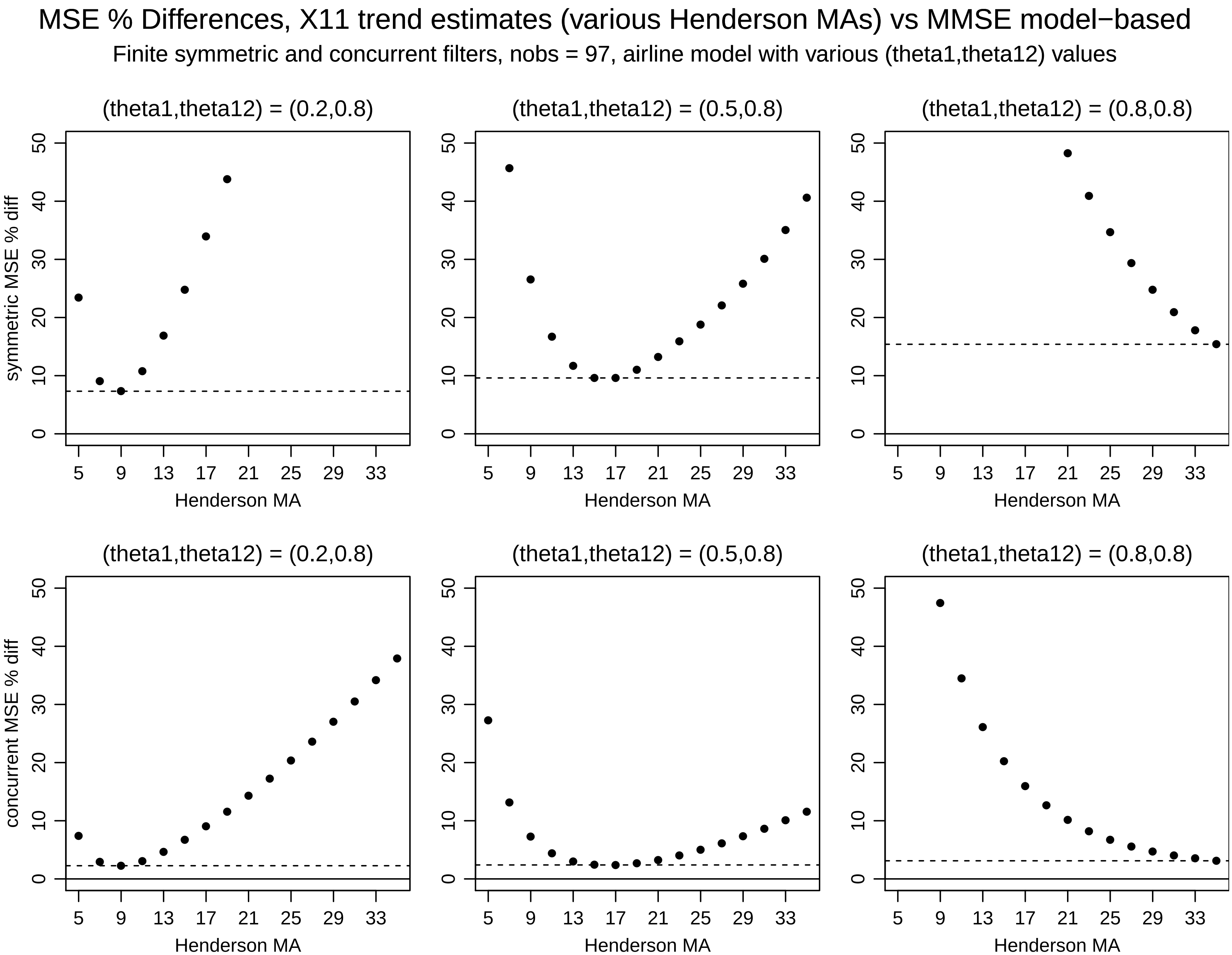
MSE % Differences, X11 trend estimates (various Henderson MAs) vs MMSE model−based
Finite symmetric and concurrent filters, nobs = 97, airline model with various (theta1,theta12) values
(theta1,theta12) = (0.2,0.8)
(theta1,theta12) = (0.5,0.8)
(theta1,theta12) = (0.8,0.8)
symmetric MSE % diff
concurrent MSE % diff
Henderson MA
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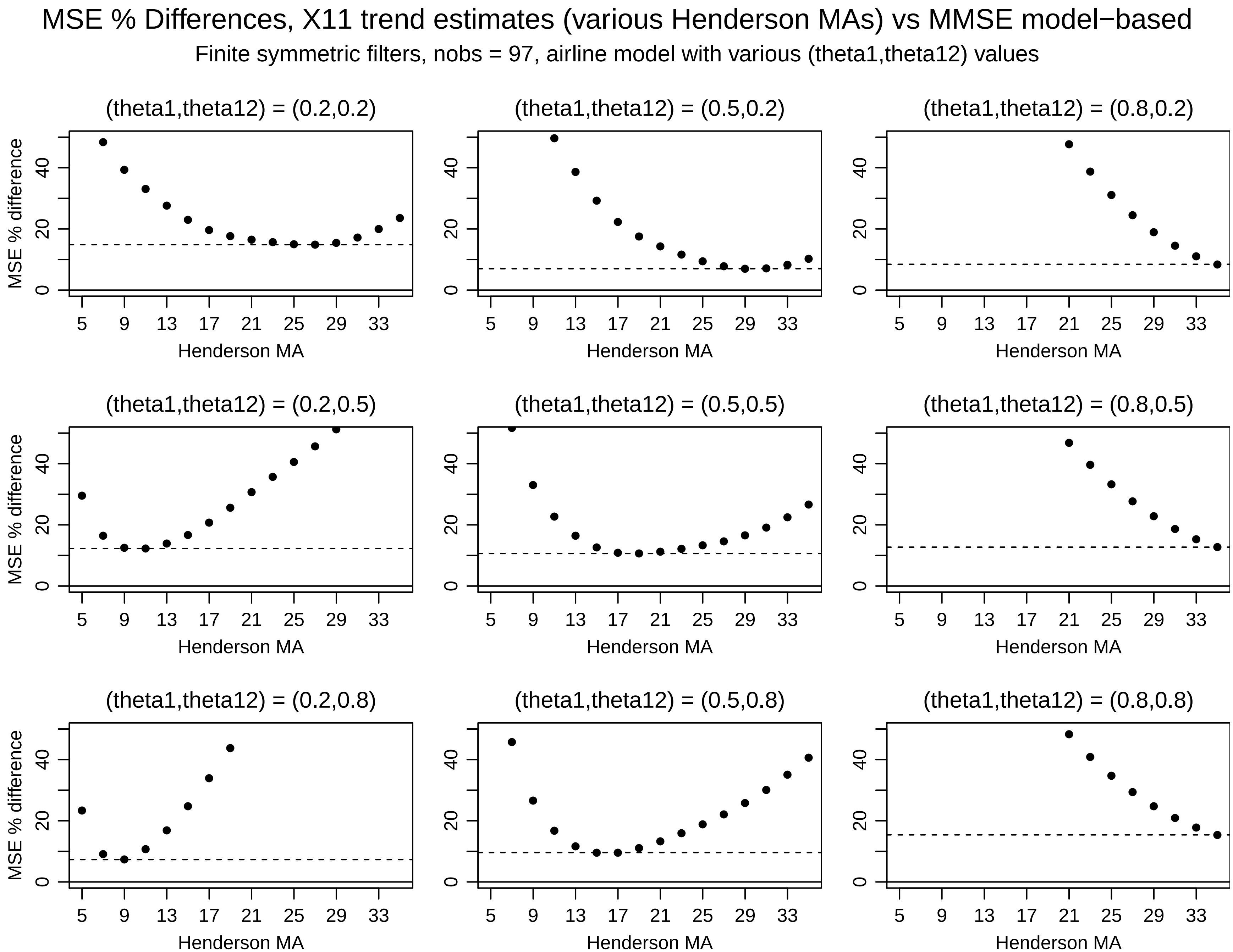
MSE % Differences, X11 trend estimates (various Henderson MAs) vs MMSE model−based
Finite symmetric filters, nobs = 97, airline model with various (theta1,theta12) values
(theta1,theta12) = (0.2,0.2)
(theta1,theta12) = (0.5,0.2)
(theta1,theta12) = (0.8,0.2)
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(theta1,theta12) = (0.8,0.5)
(theta1,theta12) = (0.2,0.8)
(theta1,theta12) = (0.5,0.8)
(theta1,theta12) = (0.8,0.8)
MSE % difference
Henderson MA
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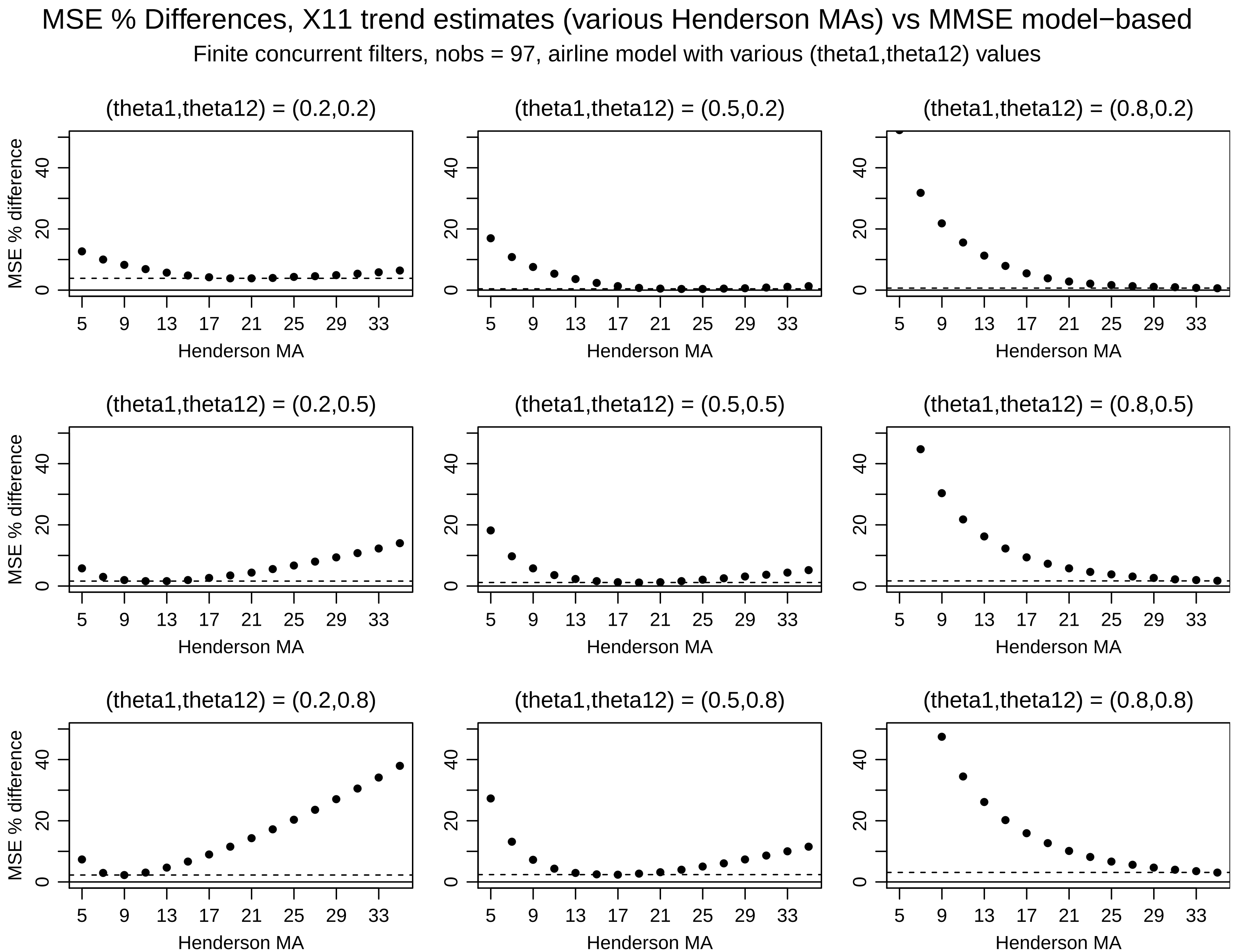
MSE % Differences, X11 trend estimates (various Henderson MAs) vs MMSE model−based
Finite concurrent filters, nobs = 97, airline model with various (theta1,theta12) values
(theta1,theta12) = (0.2,0.2)
(theta1,theta12) = (0.5,0.2)
(theta1,theta12) = (0.8,0.2)
(theta1,theta12) = (0.2,0.5)
(theta1,theta12) = (0.5,0.5)
(theta1,theta12) = (0.8,0.5)
(theta1,theta12) = (0.2,0.8)
(theta1,theta12) = (0.5,0.8)
(theta1,theta12) = (0.8,0.8)
MSE % difference
Henderson MA
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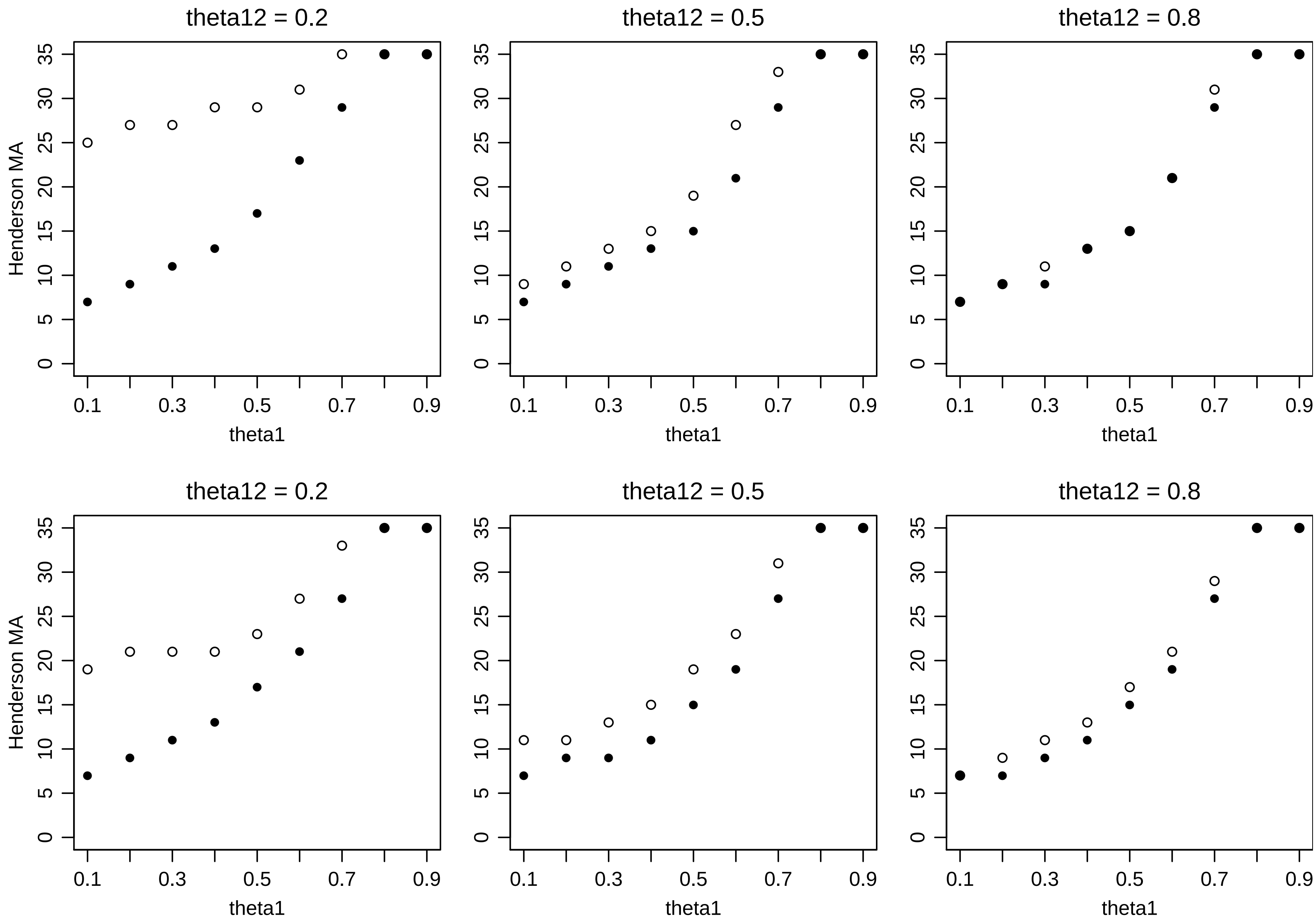
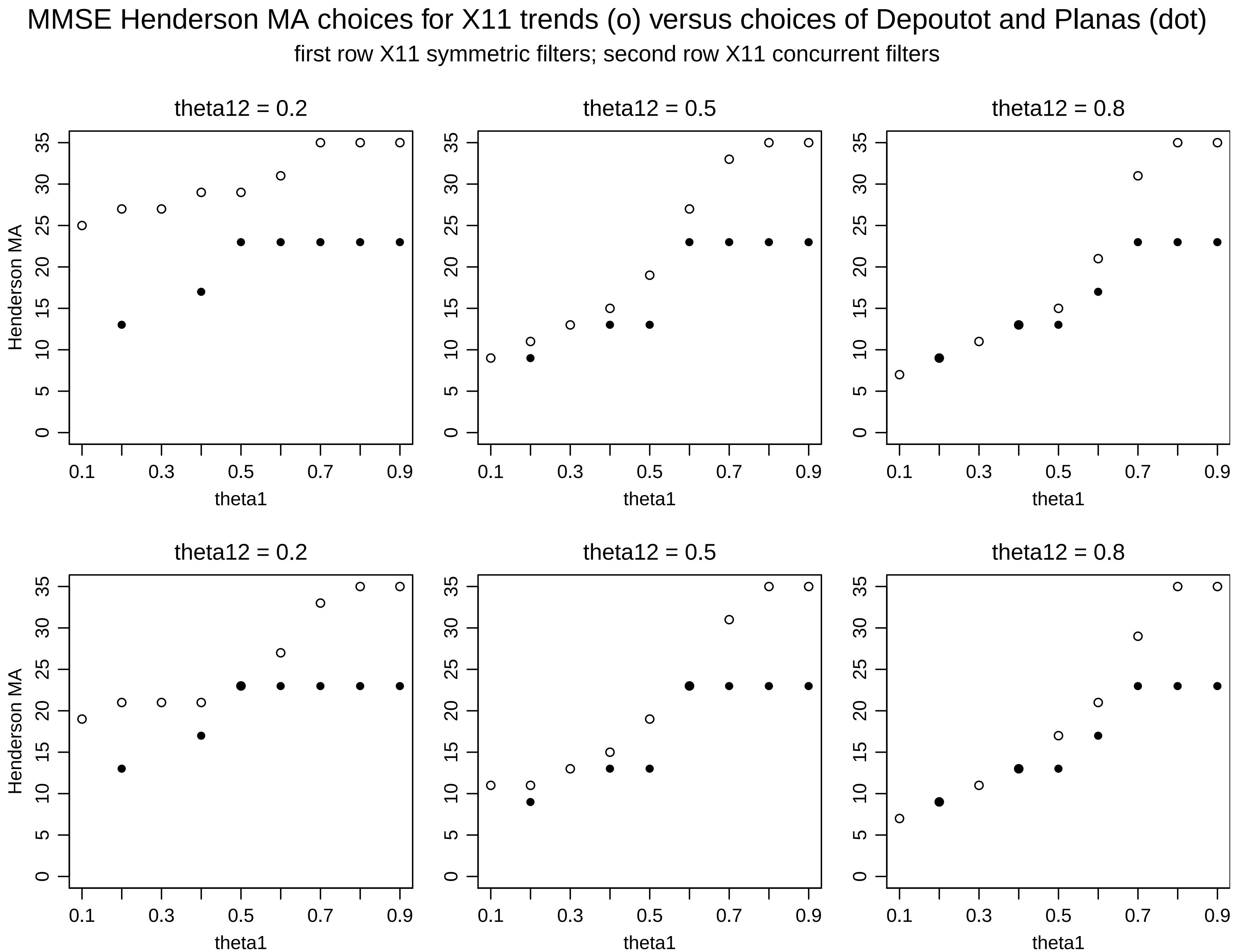
Best Henderson MA choices, seasonal + trend + noise (o) vs trend plus noise (dot) models

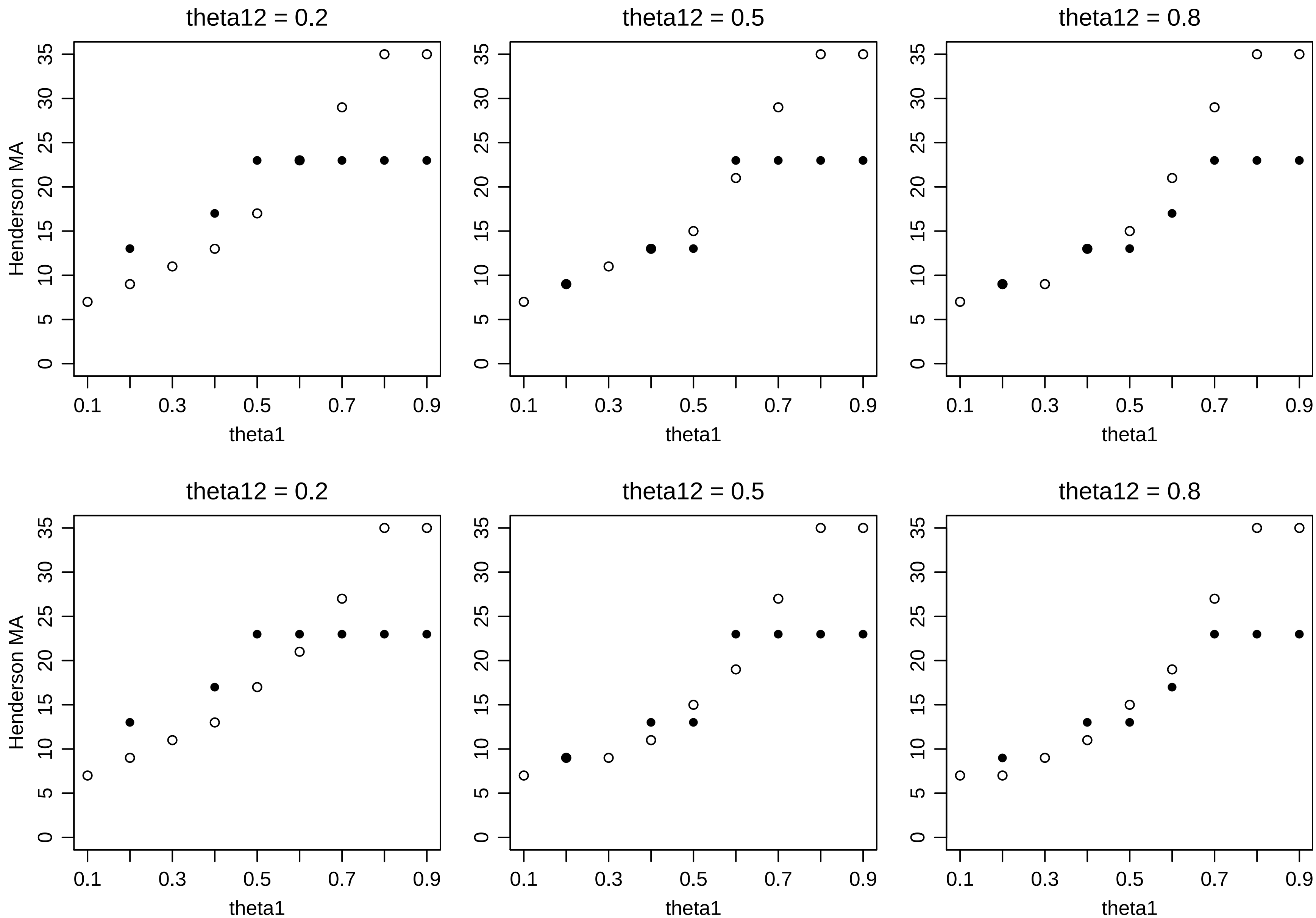
first row symmetric filters; second row concurrent filters




MMSE Henderson MA choices for X11 trends (o) versus choices of Depoutot and Planas (dot)
first row X11 symmetric filters; second row X11 concurrent filters
theta12 = 0.2
theta12 = 0.5
theta12 = 0.8
Henderson MA
theta1
0 5 10 15 20 25 30 35
0.1 0.3 0.5 0.7 0.9

MMSE Henderson MAs for trend plus noise models (o) vs choices of Depoutot and Planas (dot)

first row symmetric Henderson filters; second row concurrent Henderson filters



Conclusions

1. The best choice of Henderson MA can provide a good approximation to the MMSE estimator of a trend component that follows a trend model from the canonical decomposition of the airline model.
    - MSE increase from MMSE model-based estimator is around 10% for symmetric filters.
    - MSE increase is very small for concurrent filters.
2. Henderson MA choices close to the best (e.g., length difference of ± 2) do almost as well.
    - MA choices very distant from the best can have substantially larger MSE.
3. Similar results were seen, in most cases, from the seasonal and nonseasonal models.
    - Exceptions occurred for small values of θ_{12}.
4. Longer Henderson MAs are better for larger values of θ_1.
5. The value of θ_{12} typically matters little to the best choice of Henderson MA (with exceptions as noted in conclusion 3).