

Adapting the seasonality tests to non-integer seasonal time series

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Agenda

- Brief overview of Flexible QS test
- Traditional SD test
- Adapting the SD-test
- Bootstrapping the Variance-Covariance matrix
- Distortions of null distribution
- Summary

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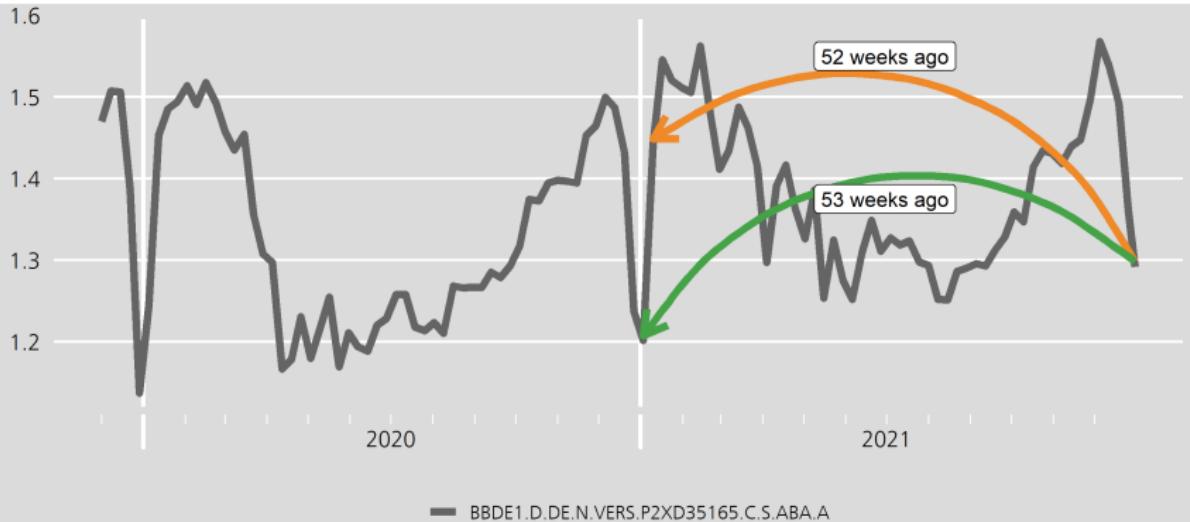
QS test

$$QS = T(T+2) \left(\frac{\hat{\rho}(y_t, y_{t-\tau})^2}{T - \tau} + \frac{[\max\{0, \hat{\rho}(y_t, y_{t-2\tau})\}]^2}{T - 2\tau} \right) \mathbb{1}_{\{\hat{\rho}(y_t, y_{t-\tau}) > 0\}}$$

Adapting the test statistic

Weekly electricity consumption in Germany

TW/h



Source: Federal Network Agency
Deutsche Bundesbank

I Adapting the test statistic

$$y_{t-\tau}^* = \alpha y_{t-\lfloor \tau \rfloor} + (1 - \alpha) y_{t-\lceil \tau \rceil}$$

I Adapting the test statistic

$$\begin{aligned}y_{t-\tau}^{\star} &= \alpha y_{t-\lfloor\tau\rfloor} + (1 - \alpha) y_{t-\lceil\tau\rceil} \\&= \alpha(\alpha y_{t-\tau} + (1 - \alpha)\varepsilon) + (1 - \alpha)([1 - \alpha]y_{t-\tau} + \alpha\varepsilon)\end{aligned}$$

Adapting the test statistic

$$\begin{aligned}y_{t-\tau}^* &= \alpha y_{t-\lfloor\tau\rfloor} + (1 - \alpha) y_{t-\lceil\tau\rceil} \\&= \alpha(\alpha y_{t-\tau} + (1 - \alpha)\varepsilon) + (1 - \alpha)([1 - \alpha]y_{t-\tau} + \alpha\varepsilon)\end{aligned}$$

with

$$\alpha = \lceil\tau\rceil - \tau.$$

I Adapting the test statistic

Adaptation with attenuation factor:

$$QS = T(T+2) \left(\frac{\left(\frac{\hat{\rho}(y_t, y_{t-\tau}^*)}{A(\alpha)} \right)^2}{T - \tau} + \frac{\left[\max \{0, \frac{\hat{\rho}(y_t, y_{t-2\tau}^*)}{A(\alpha)}\} \right]^2}{T - 2\tau} \right) \mathbb{1}_{\{\hat{\rho}(y_t, y_{t-\tau}^*) > 0\}}$$

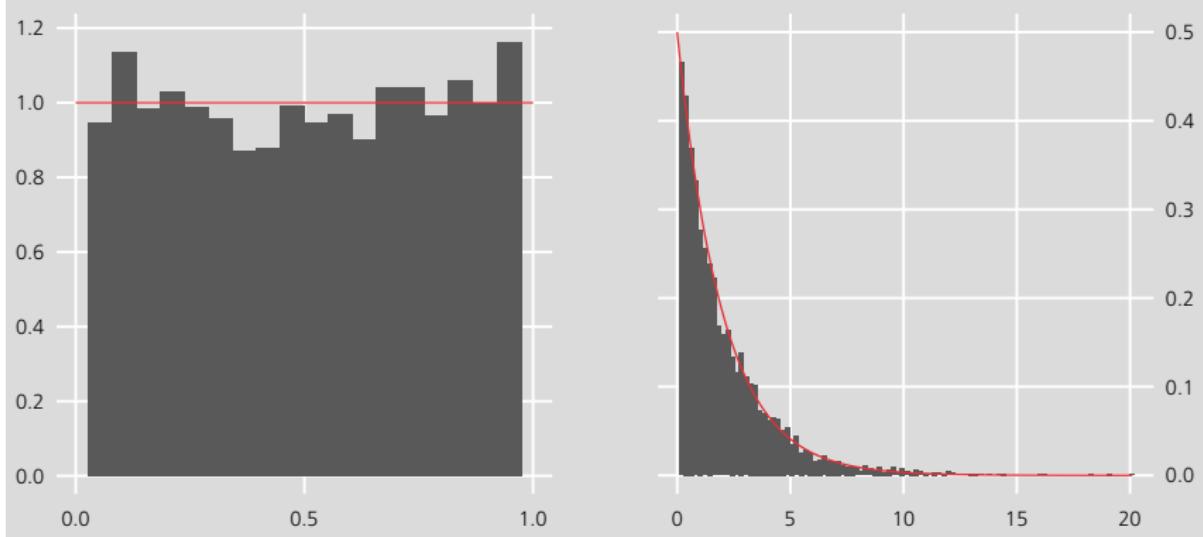
I Attenuation factor

$$A(\alpha) := \sqrt{(\alpha^2 + (1 - \alpha)^2)}$$

Null distribution for Standard QS-Test

QS-test on white noise series

Number of series = 5398 , frequency = 7 , series length = 130



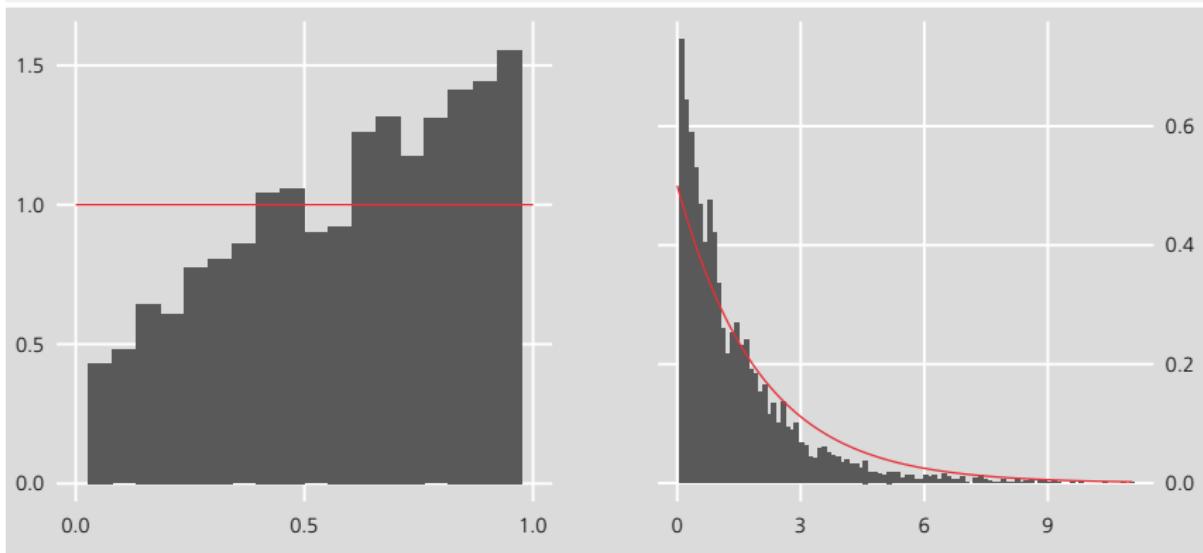
Left: P-Values with line indicating uniform distribution. Right: Test-statistic with line indicating Chi²-distribution
Deutsche Bundesbank

Null distribution for Flexible QS-Test

Without attenuation factor

Flexible QS-test (w/o attenuation) on white noise series

Number of series = 5195 , frequency = 6.52 , series length = 130



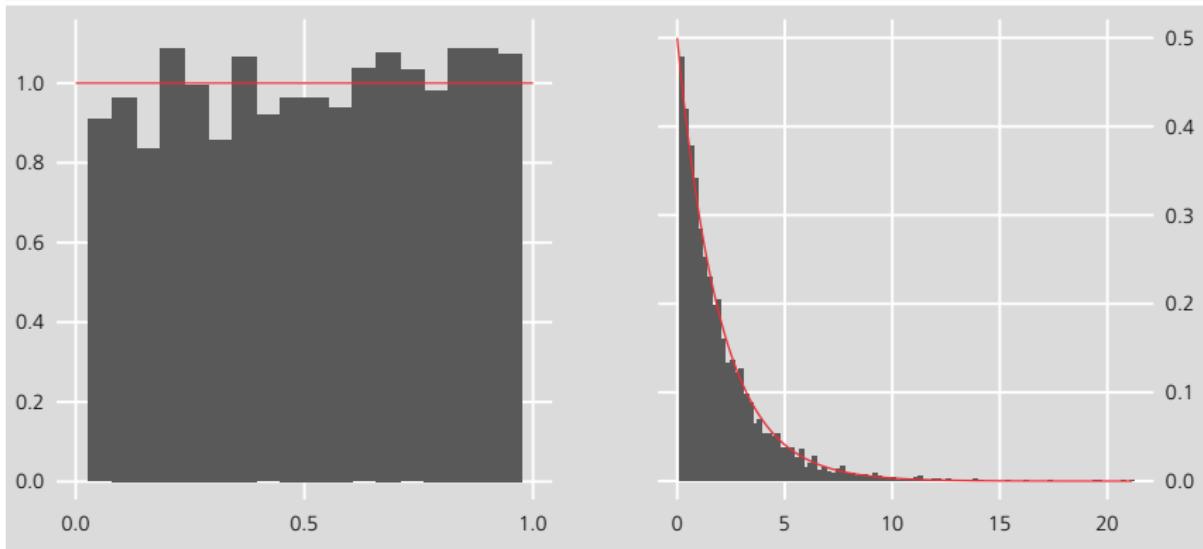
Left: P-Values with line indicating uniform distribution. Right: Test-statistic with line indicating Chi²-distribution
Deutsche Bundesbank

Null distribution for Flexible QS-Test

With attenuation factor

Flexible QS-test on white noise series

Number of series = 5205 , frequency = 6.52 , series length = 130



Left: P-Values with line indicating uniform distribution. Right: Test-statistic with line indicating Chi²-distribution
Deutsche Bundesbank

I Simulation design

- Traditional: QS test on nearest frequency
- Flexible: Adapted QS test assuming $\chi^2(2)$ -distribution

Accuracy for seasonal time series

Preliminary results

Correct classification in %, $\alpha = 0.05$

| Frequency | n | QS test | |
|-----------|------|-------------|----------|
| | | Traditional | Flexible |
| 2.33 | 1344 | 0.2 | 47.5 |
| 3.50 | 1344 | 0.1 | 79.8 |
| 6.52 | 1344 | 9.5 | 83.2 |
| 26.07 | 672 | 92.7 | 92.7 |
| 52.14 | 672 | 95.4 | 95.5 |

Size estimations for non-seasonal time series

Preliminary results

Rejection rate (in %) of H_0 at $\alpha = 0.05$

| Frequency | n | QS test | |
|-----------|-----|-------------|----------|
| | | Traditional | Flexible |
| 2.33 | 560 | 2.1 | 2.0 |
| 3.50 | 560 | 2.1 | 2.3 |
| 6.52 | 560 | 1.8 | 2.1 |
| 26.07 | 280 | 0.7 | 0.7 |
| 52.14 | 280 | 3.2 | 2.1 |

I Take-aways for QS-test

Contributions

- Generalising QS test to non-integer seasonal time series
- Deriving attenuation factor

Results of simulation

- Similar size distortions
- Usually better detection of seasonal time series

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| SD test

RegARIMA model:

$$\phi_p(B)(1 - B)^d \left(z_t - \sum_{k=1}^K \beta_k X_{k,t} \right) = \mu + \theta_q(B) \varepsilon_t$$

I SD test

Joint significance test of $K = \tau - 1$ seasonal dummies

$X_{k,t} = D_{ij,t}$ with

$$D_{ij,t} = \begin{cases} 1, & t \text{ in period } i \\ 0, & \text{otherwise.} \end{cases}$$

I SD test

The test statistic is given by

$$SD = \frac{\hat{\beta}^\top \hat{\Sigma}_{\hat{\beta}}^{-1} \hat{\beta}}{\tau - 1} \cdot \frac{T - d - p - q - \tau - 1}{T - d - p - q} \sim F(\tau - 1, T - d - p - q - \tau - 1)$$

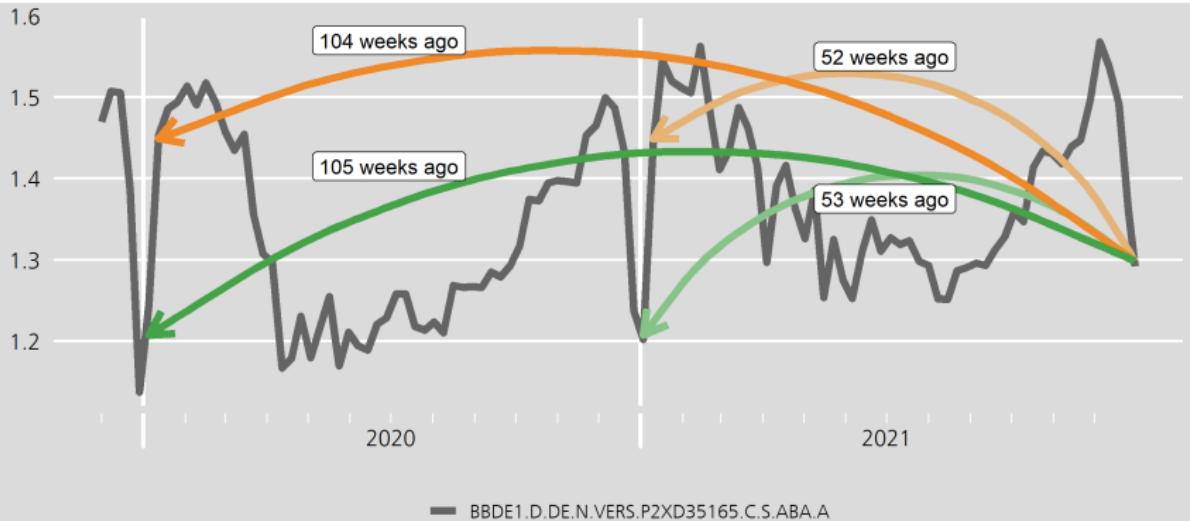
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Adapting the test statistic

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Adapting the test statistic

Generalised seasonal regressors

$$R_{i_j^*, t} = \begin{cases} 1 - |t - i_j^*|, & t \text{ neighbour to quasi-period } i_j^* \\ 0, & \text{otherwise.} \end{cases}$$

with i_j^* in $(0 - j, \tau - j, 2\tau - j, \dots)$ and j in $(0, 1, \dots, [\tau])$.

Adapting the test statistic

Generalised seasonal regressors

Example for $\tau = 2\frac{1}{3}$

| t | Regressors | |
|---|------------|---------|
| | $j = 0$ | $j = 1$ |
| 0 | 1.00 | 0.00 |
| 1 | 0.00 | 0.67 |
| 2 | 0.67 | 0.33 |
| 3 | 0.33 | 0.33 |
| 4 | 0.33 | 0.67 |
| 5 | 0.67 | 0.00 |
| 6 | 0.00 | 1.00 |
| 7 | 1.00 | 0.00 |
| 8 | 0.00 | 0.67 |
| 9 | 0.67 | 0.33 |

Adapting the test statistic

Trigonometric regressors

$$H_{j,t} = \left\{ \sin\left(\frac{2\pi j G(t)}{\tau}\right), \cos\left(\frac{2\pi j G(t)}{\tau}\right) \right\}$$

where $G(t)$ is a step function running through $1, \dots, T$.

I SD test revisited

The test statistic is given by

$$SD = \frac{\hat{\beta}^\top \hat{\Sigma}_{\hat{\beta}}^{-1} \hat{\beta}}{\tau - 1} \cdot \frac{T - d - p - q - \tau - 1}{T - d - p - q} \sim F(\tau - 1, T - d - p - q - \tau - 1)$$

I SD test revisited

The test statistic is given by

$$SD = \frac{\hat{\beta}^T \hat{\Sigma}_{\hat{\beta}}^{-1} \hat{\beta}}{\tau - 1} \cdot \frac{T - d - p - q - \tau - 1}{T - d - p - q} \sim F(\tau - 1, T - d - p - q - \tau - 1)$$

Using non-integer frequency adapted regressors, from measurement error theory (Fuller, 1987):

- $\hat{\beta}$ unbiased
- $\hat{\Sigma}_{\hat{\beta}}$ biased

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I Sieve-type bootstrap

Algorithm 1 Bootstrap $\hat{\Sigma}_{\hat{\beta}}$

- 1: Estimate RegARIMA-model of order ($p = p_0, d = d_0, q = q_0$) with seasonal regressors
 - 2: Compute model residuals $\hat{E} := \{\hat{\epsilon}_t\}_{t=1}^T$
 - 3: Center $\hat{E}^{\text{scaled}} := \hat{E} - \bar{\hat{E}}$
 - 4: **for** ($j = 1, \dots, B$) **do**
 - 5: Resample \hat{E}^{scaled} with replacement, yielding \hat{E}^{boot}
 - 6: Simulate time series from the RegARIMA(p_0, d_0, q_0) using \hat{E}^{boot} as innovations, yielding Y^{boot}
 - 7: Estimate RegARIMA(p_0, d_0, q_0) model on Y^{boot}
 - 8: Extract and save $\tilde{\beta}_j$, the coefficients of the seasonal regressors
 - 9: **end for**
 - 10: $\hat{\Sigma}_{\hat{\beta}}^{\text{boot}} := \text{Cov}(\tilde{\beta})$
-

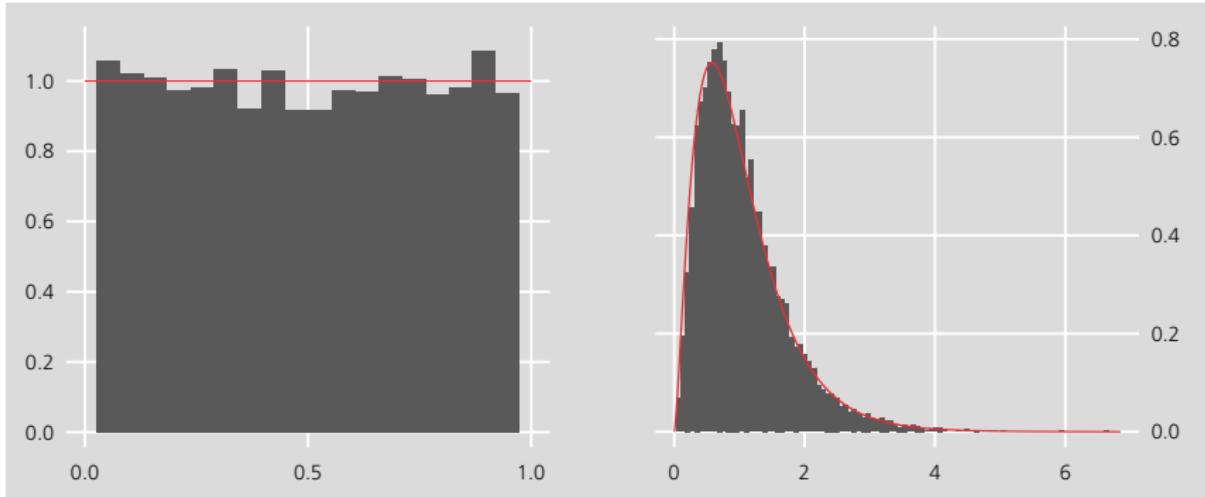
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Null distribution for Standard SD-Test

SD-test on white noise series

Number of series = 10000 , frequency = 7 , series length = 65



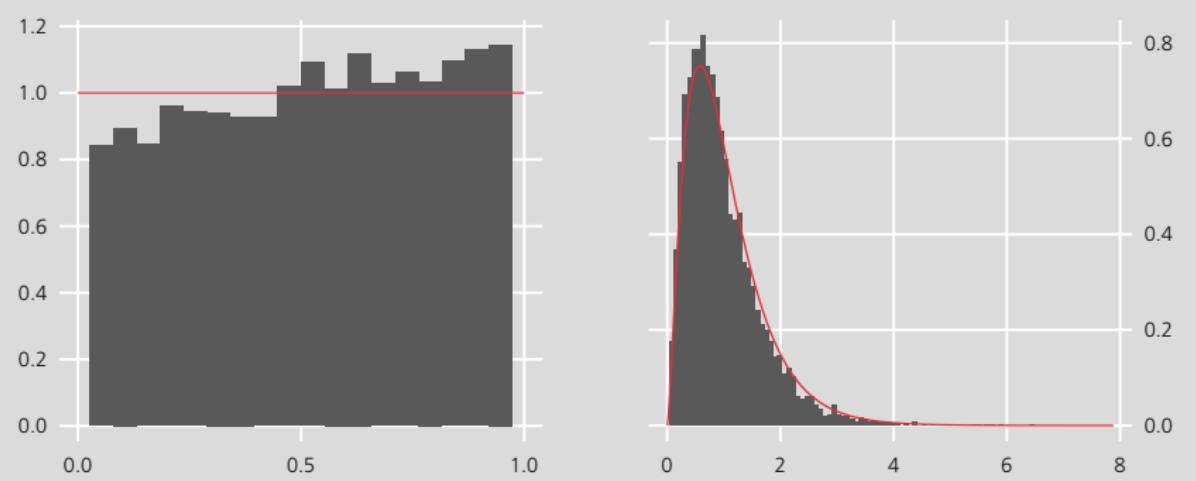
Left: P-Values with line indicating uniform distribution. Right: Test-statistic with line indicating F-distribution
Deutsche Bundesbank

Null distribution for Flexible SD-Test

Without adaptation

SD-test on white noise series

Number of series = 10000 , frequency = 6.52 , series length = 65



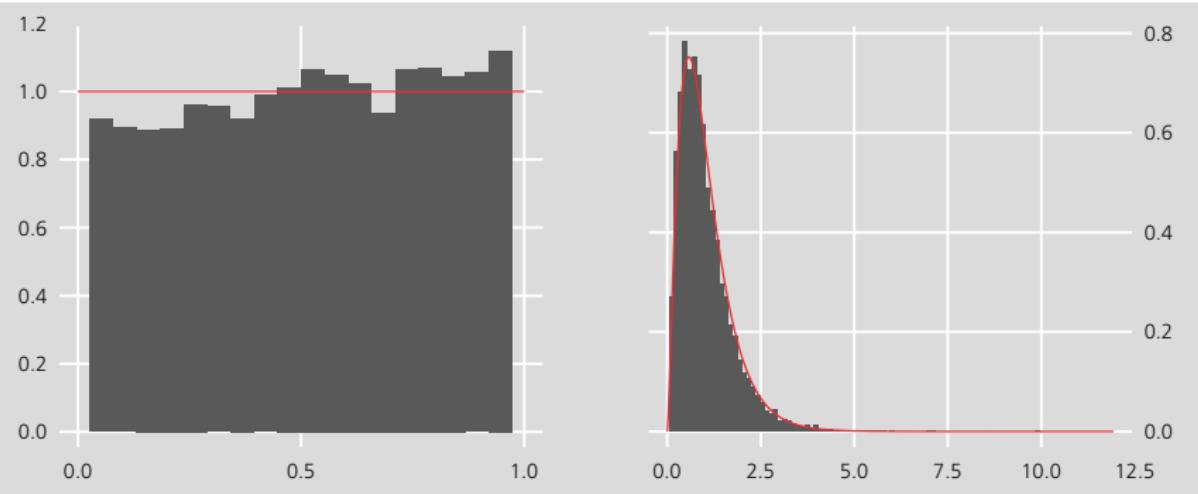
Left: P-Values with line indicating uniform distribution. Right: Test-statistic with line indicating F-distribution
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Null distribution for Flexible SD-Test

With trigonometric regressors but without adaptation

SD-test (trigonometric) on white noise series

Number of series = 10000 , frequency = 6.52 , series length = 65



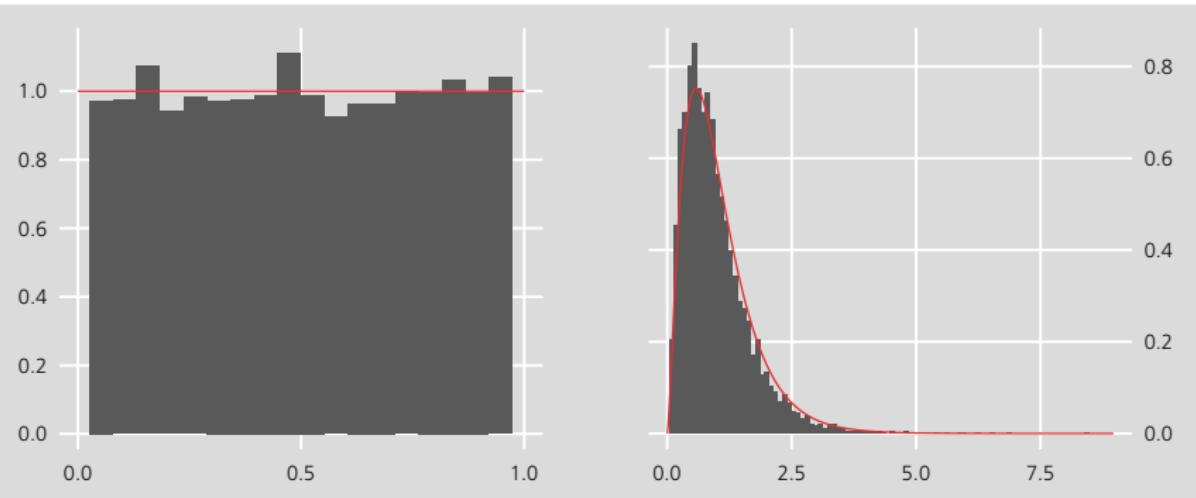
Left: P-Values with line indicating uniform distribution. Right: Test-statistic with line indicating F-distribution
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Null distribution for Flexible SD-Test

With mc-simulated null

SD-test on white noise series (simulated null)

Number of series = 10000 , frequency = 6.52 , series length = 65



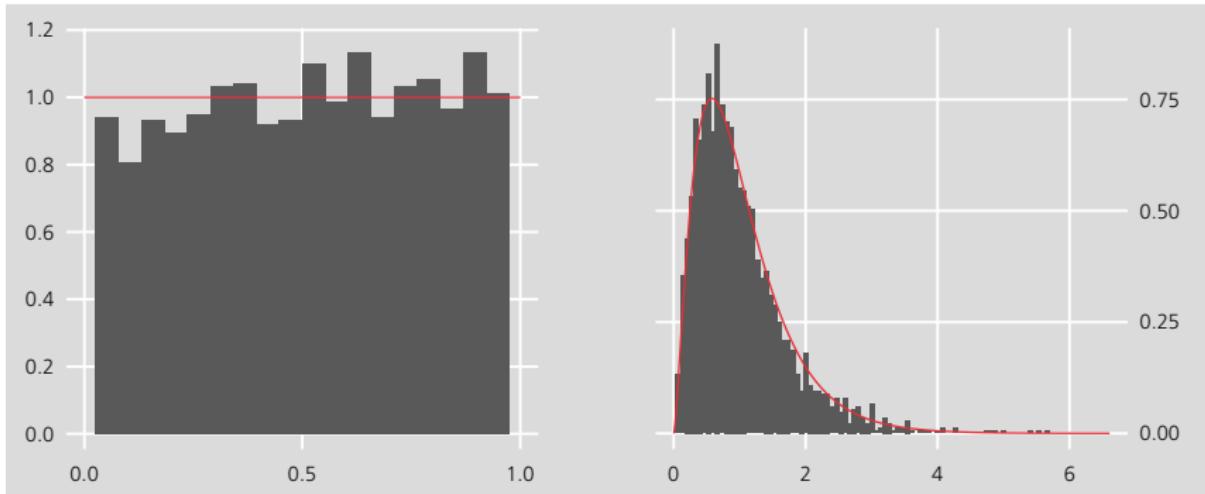
Left: P-Values with line indicating uniform distribution. Right: Test-statistic with line indicating F-distribution
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Null distribution for Flexible SD-Test

With bootstrapped $\hat{\Sigma}$

SD-test (bootstrap variant) on white noise series

Number of series = 2500 , frequency = 6.52 , series length = 65 , b = 1000



Left: P-Values with line indicating uniform distribution. Right: Test-statistic with line indicating F-distribution
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I Null distribution

- Biased null distributions even for trigonometric regressors
- Simulating null distribution helps
- Bootstrapping null distribution helps (somewhat)

Open questions

Contact me: daniel.ollech@bundesbank.de

- Why exactly does test statistic need to be adapted?
 - Even for trigonometric regressors
- Is it a small sample problem?
- How best to adapt the test statistic?

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Contributions

- Generalising SD test to non-integer seasonal time series
- Implementing bootstrap

Results of simulation

- Promising results - yet open questions

Contact

- daniel.ollech@bundesbank.de

References: Methodology

-  Fuller, W.A. (1987). Measurement error models. John Wiley & Sons, 1987.
-  Ollech, D. (2021). Seasonal Adjustment of Daily Data. Journal of Time Series Econometrics .
-  Ollech, D., Gonschorreck, N. & Hengen, L. (2021). Flexibilisation of X-11 for Higher-Frequency Data. NTTS conference 2021.