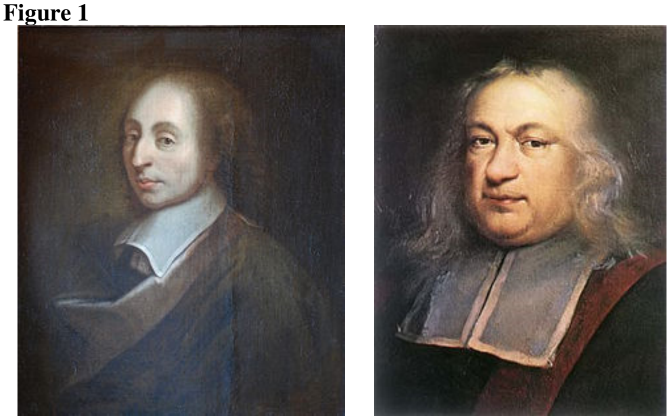
History of Probability (Part 2) - 17th Century France
The Problem of Points: Pascal, Fermat, and Huygens

Every history of probability emphasizes the correspondence between two 17th century French scholars, **Blaise Pascal** and **Pierre de Fermat**. In **1654** they exchanged letters where they discussed how to solve a particular gambling problem, now referred to as **The Problem of Points (**also called the problem of **division of the stakes)**. Simply stated, the problem is how to split the pot if the game is interrupted before someone has won. Pascal and Fermat's work on this problem led to the development of formal rules for probability calculations.

Figure 1



Blaise Pascal 1623-1662 Pierre de Fermat 1601-1665

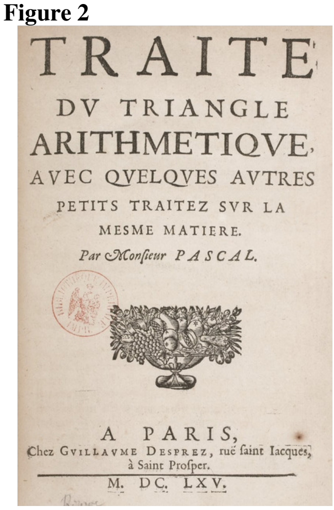
The problem of points concerns a game of chance with two competing players who have equal chances of winning each round. [Imagine that one round is a toss of a coin. Player A has heads, B has tails.] The winner is the first one who wins a certain number of pre-agreed upon rounds. [Suppose, for example, they agree that the first person to win 3 tosses wins the game.] Whoever wins the game gets the whole pot. [Say, each player puts in $6, so the total pot is $12. If A gets three heads before B gets three tails, A wins $12.] Now suppose that the game is interrupted before either player has won. How does one then divide the pot fairly? [Suppose they have to stop early and at that point A has won two tosses and B has won one.] Should each get $6, or should A get more since A was ahead when they stopped? What is fair?

Historically, it is important to note that all of these gambling problems were framed in terms of odds for winning a game and not in terms of probabilities of individual events. A gambler wanted to know how much money from each player was fair. For instance, if we figure out that the odds are 5 to 2 in my favor in some game then I should put 5 coins in the pot for every 2 of yours.

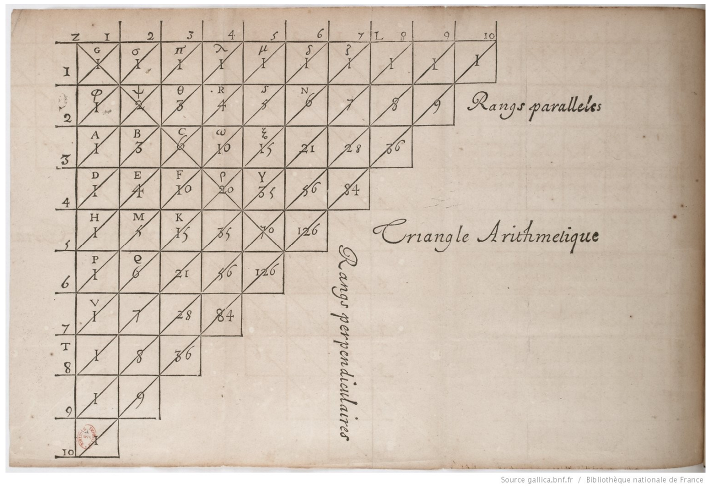
In their exchange of letters, Pascal and Fermat came to an agreement on the solution by two different methods, but Pascal's approach led to more efficient computation. Fermat's method was essentially to lay out all possible paths to the needed number of wins and count how many went to each player. This method works but gets cumbersome as the number needed to win increases. Pascal's approach made use of his **arithmetic triangle**, which we now call **Pascal's triangle.** The triangle was known by mathematicians around the world well before this time in connection with the expansion of binomial expressions. It is named after Pascal because he wrote a small book about it called ***Traite du triangle arithmétique*** (***A treatise on the arithmetic triangle*****),** in which the triangle is applied to questions about odds.

Pascal wrote the book in French, not Latin and that actually slowed its findings spreading into other countries. He shared the material of the book at the time of the letter writing, but the book itself was not officially published until 1665, a few years after his death. Figure 2 shows the cover of the 1665 publication and the book's image of the triangle. You can see that Pascal's orientation of the table is different from the way we usually do it now.

Figure 2


TRAITE
DV TRIANGLE
ARITHMETIQVE,
AVEC QVELQVES AVTRES
PETITS TRAITEZ SVR LA
MESME MATIERE.
Par Monsieur PASCAL.

A PARIS,
Chez GVILLAVME DESPREZ, ruë saint Iacques, à Saint Prosper.
M. DC. LXV.



Why were Pascal and Fermat working on the problem of points?

Pascal's father, Etiénne, brought his children to Paris in 1631 specifically to supervise their education. Etiénne was connected to the group of philosophers, scientists, and mathematicians that met regularly at the Parisian home of the scholar/priest Marin Mersenne, and Etiénne got the OK to bring his 14 year old son, Blaise. Mersenne regularly distributed the work discussed at these meetings through letters in Latin to scholars all over Europe. The Pascal family left Paris in 1639, but Blaise continued his connection to the Paris group. For the next 20 years he was an active mathematician and physicist.

In 1652, the problem of points was known to the Mersenne group, and about that time was presented to Pascal by the "Chevalier de Méré," a nobleman, amateur mathematician, and sophisticated gambler, with whom he spent some time. Chevalier de Méré was not his real name, which was Antoine Gombaud**.** De Méré was the name of a character in books he wrote, and his friends starting calling him that. Pascal worked on the problem and sent his solution to Fermat, who was an older man, one of France's great mathematician, also well known in the Mersenne group. Then they wrote back and forth discussing it and other problems. Here is an excerpt from one of the letters that can give you the flavor of the conversation.

Pascal to Fermat
Wednesday, July 29, 1654

Monsieur, -

1. Impatience has seized me as well as it has you, and although I am still abed, I cannot refrain from telling you that I received your letter in regard to the problem of the points yesterday evening from the hands of M. Carcavi, and that I admire it more than I can tell you. I do not have the leisure to write at length, but, in a word, you have found the two divisions of the points and of the dice with perfect justice. I am thoroughly satisfied as I can no longer doubt that I was wrong, seeing the admirable accord in which I find myself with you.

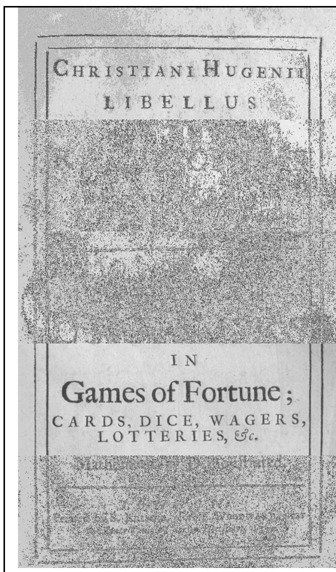
I admire your method for the problem of the points even more than that of the dice. I have seen solutions of the problem of the dice by several persons, as M. le chevalier de Mere, who proposed the question to me, and by M. Roberval also M. de Mere has never been able to find the just value of the problem of the points nor has he been able to find a method of deriving it, so that I found myself the only one who knew this proportion.

2. Your method is very sound and it is the first one that came to my mind in these researches, but because the trouble of these combinations was excessive, I found an abridgment and indeed another method that is much shorter and more neat, which I should like to tell you here in a few words; for I should like to open my heart to you henceforth if I may, so great is the pleasure I have had in our agreement. I plainly see that the truth is the same at Toulouse and at Paris.

It is fascinating to realize how slowly mathematical discoveries were spread in the 17th century. Today a mathematician or physicist can publish her work on the internet and immediately a global community of fellow researchers can see and discuss it. During the 17th century sharing involved talking face-to-face, writing letters, and printing books. A “conversation” about a new finding could easily take months. Informal scholarly groups were formed in many European locations. The Renaissance which had been primarily centered in Italy in the 16th century (Galileo, da Vinci, Cardano) expanded northward into France, Holland, Germany, and England. It was a time of great achievement in science and mathematics. The members of these informal groups exchanged letters, met in people’s homes, and visited one another when they traveled. To deal with language barriers their published work was often written or translated into Latin. Towards the end of the 17th century a number of these groups became the foundations of official government supported scientific research. For example, The French Academy of Sciences (*Académie des sciences*) was founded in 1666 by King Louis XIV. The Royal Society in England was chartered in 1660 by King Charles II. These organizations today, more than 350 years later, are among the most prestigious science organizations in the world.

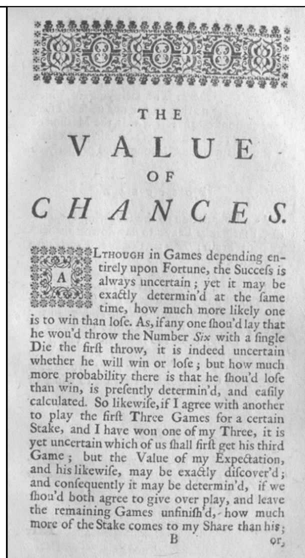
Christiaan Huygens 1629-1695

The Parisian group received visitors from outside France. A key such person for the history of probability was a young Dutch scholar, **Christiaan Huygens**, who showed up in 1665, heard about the problem of points, and became interested in the mathematics of probability. He did not meet Pascal or Fermat at that time to see what they had done, but worked on the problem himself and wrote a small booklet on the subject in 1656, which is widely considered the **first text book on probability**, titled ***De ratiociniis in ludo aleae*** (*Reasoning in Games of Chance*). In Figure 3, you can see that Huygens’ explanation of how to reason about chance was based directly on the problem of points.



CHRISTIANI HUGENII
LIBELLUS
IN
Games of Fortune;
CARDS, DICE, WAGERS, LOTTERIES, &c.

English translation 1714



THE
VALUE
OF
CHANCES.

ALTHOUGH in Games depending entirely upon Fortune, the Success is always uncertain; yet it may be exactly determin'd at the same time, how much more likely one is to win than lose. As, if any one shou'd lay that he wou'd throw the Number *Six* with a single Die the first throw, it is indeed uncertain whether he will win or lose; but how much more probability there is that he shou'd lose than win, is presently determin'd, and easily calculated. So likewise, if I agree with another to play the first Three Games for a certain Stake, and I have won one of my Three, it is yet uncertain which of us shall first get his third Game; but the Value of my Expectation, and his likewise, may be exactly discover'd; and consequently it may be determin'd, if we shou'd both agree to give over play, and leave the remaining Games unfinish'd, how much more of the Stake comes to my Share than his;

B or,

The Value of Chances

ALTHOUGH in Games depending entirely upon Fortune, the Success is always uncertain; yet it may be exactly determin'd at the same time, how much more likely one is to win than lose. As, if any one shou’d lay that he wou’d throw the Number Six with a single die the first throw, it is indeed uncertain whether he will win or lose; but how much more probability there is that he shou’d lose than win, is presently determin’d, and easily calculated. So likewise, if I agree with another to play the first Three Games for a certain Stake, and I have won one of my Three, it is yet uncertain which of us shall first get his third Game; but the Value of my Expectation and his likewise, may be exactly discovered; and consequently it may he determin’d, if we shou’d both agree to give over play, and leave the remaining Games unfinish'd, how much more of the Stake comes to my Share than his; or, if another desired to purchase my Place and Chance, how much I might just sell it for. And from hence an infinite Number of Questions may arise between two, three, four, or more Gameters: The satisfying of which being a thing neither vulgar nor useless, I shall here demonstrate in few words, the Method of doing it; and then likewise explain particularly the Chances that belong more properly to Dice.

Huygens wrote his piece in Dutch. His mentor, Franz van Schooten, translated it into Latin and attached it as an index to his own large and widely used more general math book, *Exercitationes mathematicae libri quinque* (*Five Books of Mathematical Exercises*) published in 1667. Huygens’s contribution to probability thus became known, and his methods of solving problems became standard. It was the basic text for the mathematical analysis of chance for the next 50 years.

Exercises:

1. For the problem of points given on page 1, what do you think is a fair way to split the $12? Why? Try to justify your decision mathematically. What is the probability that A would have won had the game continued? That B would have won?

2. The following paragraphs are a continuation of Pascal's letter to Fermat shown earlier. Pascal gives his solution. Work out the problem yourself and see if you agree with Pascal's 48 to 16 split of the 64 pistoles pot. (A pistole was a gold coin in common European use in the 17th century.)

> This is the way I go about it to know the value of each of the shares when two gamblers play, for example, in three throws, and when each has put 32 pistoles at stake:
>
> Let us suppose that the first of them has two (points) and the other one. They now play one throw of which the chances are such that if the first wins, he will win the entire wager that is at stake, that is to say 64 pistoles. If the other wins, they will be two to two and in consequence, if they wish to separate, it follows that each will take back his wager that is to say 32 pistoles.
>
> Consider then, Monsieur, that if the first wins, 64 will belong to him. If he loses, 32 will belong to him. Then if they do not wish to play this point, and separate without doing it, the first should say "I am sure of 32 pistoles, for even a loss gives them to me. As for the 32 others, perhaps I will have them and perhaps you will have them, the risk is equal. Therefore let us divide the 32 pistoles in half, and give me the 32 of which I am certain besides." He will then have 48 pistoles and the other will have 16.

3. Suppose there were three players, and the game was halted with player A having won two rounds, player B having won one round, and player C having won one round at that point. How would you split the pot fairly if the three players had each put in 54 pistoles? Assume you need to win three rounds to win the game. Assume that at each round each player has the same chance of winning.

4. Exercise: Refer to the picture of Pascal's triangle shown earlier.
Pascal describes how to construct his triangle as shown below. Use his system to reproduce the table shown from Pascal's book.

Start with the top row made of all 1s. Then:

In every arithmetical triangle, each cell is equal to the sum of all the cells of the preceding parallel row, from its own perpendicular row to the first, inclusive.

Conſequence ſeconde.

En tout Triangle Arithmetique, chaque cellule eſt égale à la ſomme de toutes celles du rang parallele precedent, compriſes depuis ſon rang perpendiculaire iuſques au premier incluſiuement.

So, to get the second row we will add entries from the first row.

The first cell in the second row is 1+0 = 1, because there is nothing preceding the 1 in the row above that cell. The second cell in the second row is 1+1 = 2. The third cell is in the second row is 1+1+1 = 3, etc. To get the third row, you add consecutive values from the second row.

Sources: *Willful Ignorance*, Herbert Weisberg, Wiley;
Games, Gods & Gambling, F.N. David,
Pascal Fermat Correspondence: http://www.york.ac.uk/depts/maths/histstat/pascal.pdf
Photos of mathematicians: Wikipedia.
Pascal's Book: http://gallica.bnf.fr/ark:/12148/btv1b86262012/f7.image
Huygens' Book: http://posner.library.cmu.edu/Posner/books/book.cgi?call=519_H98L_1714