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Frequency of Discharges From Ungaged Catchments

J. E. Nash

Abstract—A method is presented by which the frequency of discharges from ungaged catchments can be related to the rainfall frequency and the unit hydrograph for the catchment. The application of the method to well drained catchments in Ireland is demonstrated.

Introduction—A method sometimes used to calculate the frequencies of discharges from an ungaged catchment, being given the unit hydrograph and the rainfall frequency data, is to assume that the frequency of any given discharge is the frequency of the amount of rainfall required, in a certain critical period, to produce the given discharge. In the notation defined below, this assumption may be stated as

$$F_q = F\left[\frac{q}{U(T_c, P)P}, T_c \text{ or less}, A\right] \quad (1)$$

where T_c is the critical period for the catchment. The critical period may have a physical meaning, as the time of concentration of the catchment, or it may be defined as the value of T which makes

$$F\left[\frac{q}{U(T, P)P}, T \text{ or less}, A\right]$$

a maximum. The author suggests that, even when T_c is given the latter definition, Eq. (1) underestimates F_q for the following reasons: (1) The frequencies of storms of duration greater than T_c, which produce q but which contain no period T_c of sufficient rainfall to produce q are neglected. (2) For storms of duration less than T_c the peak of the unit hydrograph is greater than the peak of the unit hydrograph of period T_c. Consequently, in such cases, q can be produced by storms of lesser amount than $q/U(T_c, P)P$

The principle presented in this paper is that the frequency of q can be obtained by adding the frequencies of all storms which produce q. That is, the frequency of q is the sum of the frequency of the amount of rainfall required to produce q, in a storm of duration one hour or less, plus the frequency of the amount of rainfall required to produce q in a storm of duration between one and two hours, plus the frequency of the amount of rainfall required to produce q in a storm of duration two to three hours, etc.

Notation—The following notation is used.

A the catchment area, sq mi

S the catchment slope, in parts per 10,000

R the amount or rainfall, inches

T the duration of rainfall, hours

h an arbitrary short period

M the mean annual rainfall, inches

P the proportion of the total rainfall which becomes effective

Q the peak discharge, ft³/sec

q the peak discharge per square mile, ft³/sec

F_Q the frequency of Q, occurrences per year

F_q the frequency of q, occurrences per year

$F(R, T \text{ or less}, A)$ the frequency with which a rainfall amount R inches in a period T hours or less, on a catchment of area A is equalled or exceeded

$F(R, T \text{ or less}, 0)$ the frequency with which a rainfall amount R inches in a period T hours or less, at a single gage, is equalled or exceeded

$F(R, T, A)$ the frequency with which a rainfall amount R inches in a period of duration between $T \pm h$ on a catchment of area A is equalled or exceeded

$F(R, T, 0)$ the frequency with which a rainfall amount R inches in a period of duration between $T \pm h$, at a single gage, is equalled or exceeded

$\mu(T, A)$ a function of duration and area, giving the ratio of the frequency of any amount in a period of duration between $T \pm h$ on a catchment of area A to the frequency of the same amount in the same period at a single gage, that is

$$\mu(T, A) = \frac{F(R, T, A)}{F(R, T, 0)}$$

$U(T, t)$ the ordinate at time t, of the unit hydrograph of period T, ft³/sec. per square mile for one inch of effective rainfall

$U(T, P)$ The peak of the unit hydrograph of period T

$U(0, t)$ the ordinate at time t, of the instantaneous unit hydrograph

$U(0, P)$ the peak of the instantaneous unit hydrograph

V the area of the unit hydrograph; that is 1 inch × 1 square mile = 645 ft³ hr/sec

y the ordinate of the dimensionless unit hydrograph $= U(0, t)/U(0, P)$

x the abscissa of the dimensionless unit hydrograph $= tU(0, P)/V$

S_t the ordinate, at time t, of the S curve

Assumptions—(1) On a given catchment P is constant. This assumption is not in accordance with the infiltration theory but it is a simplification usually found necessary in developing runoff formulas. It does not create any theoretical difficulty, as on a completely impervious catchment, where the runoff is always 100 pct, the assumption is exact.

(2) On a given catchment, a rainfall amount R in a period of duration between $T \pm h$ causes a peak discharge q, which is proportional to R and is a function of T in accordance with the unit hydrograph theory; that is

$$q = PRU(T, P) \tag{2}$$

This assumption contains a possibly serious source of error which may be demonstrated as follows; Figure 1 shows a non uniform rain storm in which the amount of rainfall R_1, in the period T_1, is such as causes q in accordance with Eq. (2). Similarly the amount R_2 in the period T_2 is sufficient to cause q. In accordance with our assumption, we take this storm as causing two peak discharges of q, whereas in fact one peak discharge greater than q occurs.

(3) On a given catchment the frequency of q is the sum of the frequencies of all storms which cause q; that is

$$F_q = \Sigma F(R, T, A) \quad \text{for } T = 0, 2h, 4h, 6h \cdots \tag{3}$$

where $R = q/U(T, P)P$

(4) That we have an equation, $F(R, T$ or less, $A)$ applicable to the region in which the catchment lies, giving the frequency of rainfall amounts on a catchment of area A in a given period or less as a function of the area and period. While in most regions, data are available relating amount-duration-frequency for rainfall at single gages, there is a great scarcity of information on the manner in which the frequency of a given amount in a given period diminishes as the area over which the amount is measured increases.

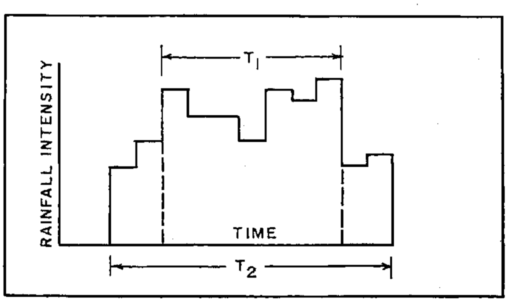

FIG. 1 – Non uniform rainfall

In the general theory below, two formulas are assumed, the first, $F(R, T$ or less, $0)$, giving the frequency of an amount R in period T or less, at a single gage, the second $\mu(T, A)$ giving the ratio of the frequency of any amount, in a period of duration between $T \pm h$ on a catchment of area A, to the frequency of the same amount in the same period at a single gage; that is

$$\mu(T, A) = \frac{F(R, T, A)}{F(R, T, 0)} \tag{4}$$

The general theory—

$$F(R, T, 0) = F(R, \overline{T + h} \text{ or less}, 0) - F(R, \overline{T - h} \text{ or less}, 0) \tag{5}$$

for all values of T not less than h

By Assumption 3.

$$F_q = \Sigma F\left[\frac{q}{PU(T, P)}, T, A\right] \quad \text{for } T = 0, 2h, 4h, 6h \text{ etc.}$$

$$= \mu(0, A)F\left[\frac{q}{PU(0, P)}, h \text{ or less}, 0\right] + \Sigma\, \mu(T, A)F\left[\frac{q}{PU(T, P)}, T, 0\right] \tag{6}$$

for $T = 2h, 4h, 6h$ etc., which by (5) becomes

$$F_q = \mu(0, A)F\left(\frac{q}{PU(0, P)}, h \text{ or less}, 0\right) + \Sigma\mu(T, A)\left\{F\left(\frac{q}{PU(T, P)}, \overline{T + h} \text{ or less}, 0\right) - F\left(\frac{q}{PU(T, P)}, \overline{T - h} \text{ or less}, 0\right)\right\} \tag{7}$$

for $T = 2h, 4h, 6h \cdots$

This equation is general and can be made applicable to any region by inserting the local expressions for $F(R, T$ or less, $0)$, $\mu(T, A)$, $U(0, P)$ and $U(T, P)$. Normally, one would express $U(T, P)$ as a function of $U(0, P)$ and T.

The application of equation 7 to well drained catchments in Ireland—For $F(R, T$ or less, $0)$, the most suitable rainfall data, available in Ireland, are *Dillon's* [1954] analysis of a 35 years record of a continuous recording rain gage at Cork. Dillon's analysis gives $F(R, T$ or less, $0) = CT^2/R^5$ where C is a constant having the value 0.036 at Cork. The author uses Dillon's formula for the whole of Ireland, adjusting C in proportion the fifth power of the mean annual rainfall in the region in which the catchment lies. Thus the formula becomes

$$F(R, T \text{ or less}, 0) = 0.036(M/40)^5T^2/R^5 \quad (8)$$

where M is the mean annual rainfall in inches (= 40 at Cork).

For $\mu(T, A)$, as far as the author is aware, no data are available in Ireland relating the frequencies of equal rainfall amounts on different areas. However, the late George *Bramsby-Williams* [1952] prepared a set of curves for the British Isles, showing the amounts of rainfall in various periods to be expected with equal frequency on catchments of various areas. Figure 2 was obtained from these curves. It is plotted on logarithmic-probability paper to obtain greater accuracy in reading higher values of μ.

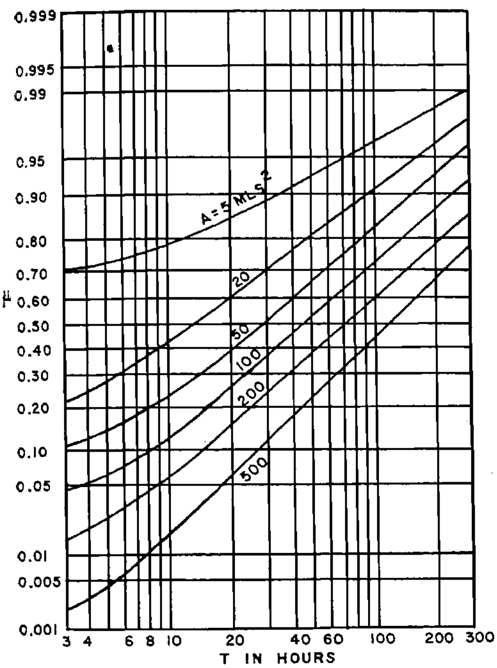

FIG. 2 – Variation of mean rainfall intensity with area and duration

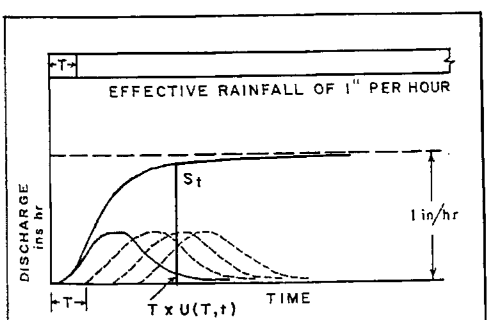

FIG. 3 – Obtaining the S curve from a known unit hydrograph

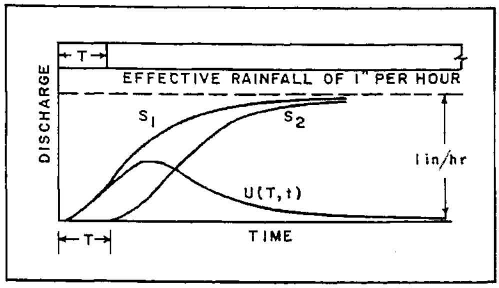

FIG. 4 – Obtaining a unit hydrograph of any period from a known S curve

For $U(0, P)$, a correlation, between the instantaneous unit hydrograph and the catchment characteristics, has been obtained by the staff of the Office of Public Works, Ireland. This correlation, which is applicable only to catchments containing no large lakes or flooded areas gives

$$U(0, P) = 4.17A^{-0.25}S^{0.6425} \quad (9)$$

where S is obtained by imposing a grid on a map of the catchment and taking the median value of the maximum slope occurring at the intersections of the grid. The coefficient of correlation for the nine points used in the correlation was 0.75.

For $U(T, P)$, the relation between $U(T, P)$ and $U(0, P)$ is a function of the shape of the instantaneous unit hydrograph. In Figure 3 the uniform continuous effective rainfall of intensity one inch per hour, causes the hydrograph (S curve) to rise as shown and approach a value equal to the intensity of the effective rainfall asymptotically. The amount of effective rainfall in the first period T is T inches and hence the discharge, at time t, due to this rainfall, is $TU(T, t)$. Similarly the discharge, at time t, due to the second rainfall period, is $TU(T, t - T)$ and the discharge at time t, due to the third rainfall period, is $TU(T, t - 2T)$, etc.

Now S_t, the ordinate of the S curve at time t, equals the sum of the discharges, at time t, due to each rainfall period; that is

$$S_t = T[U(T, t) + U(T, t - T) + U(T, t - 2T), \text{etc.,}] \quad (10)$$

As T, the period of the unit hydrograph, is diminished indefinitely, the right-hand side of (10) approaches the area of the instantaneous unit hydrograph between $t = 0$ and t; that is

$$S_t = \int_0^t U(0, t)\, dt \tag{11}$$

In Figure 4, the uniform continuous effective rainfall of intensity one inch/hour causes the S curve marked S_1. S_2 is the same curve moved to the right on amount T. Now S_2 may be looked upon as resulting from the uniform continuous effective rainfall beginning T hours later than that which produced S_1. The difference between the two S curves must be due to the effective rainfall from 0 to T. The volume of this effective rainfall is T inches. Therefore the unit hydrograph of period T is the difference between the two S curves divided by T.

$$\begin{aligned} U(T, t) &= (1/T)(S_t - S_{t-T}) \\ &= (1/T)\left[\int_0^t U(0, t)dt - \int_0^{t-T} U(0, t)\, dt\right] \\ &= (1/T)\int_{t-T}^t U(0, t)\, dt \qquad (12) \end{aligned}$$

That is, the ordinate at time t of the unit hydrograph of period T is the mean ordinate of the instantaneous unit hydrograph between t and $t - T$. In particular, the peak of the unit hydrograph of period T is the mean ordinate of the instantaneous unit hydrograph between t and $t - T$, where t is chosen to obtain the maximum value of the integral.

In certain cases, (12) can be simplified further. The curve obtained by dividing the ordinates of an instantaneous unit hydrograph by the peak and multiplying the abscissae by the peak divided by the area of the instantaneous unit hydrograph is known as the dimensionless unit hydrograph. It expresses $Y = U(0, t)/U(0, P)$ as a function of $X = tU(0, P)/V$. In the correlation of the unit hydrographs with the catchment characteristics, referred to above, it was found that the dimensionless unit hydrographs derived from the several

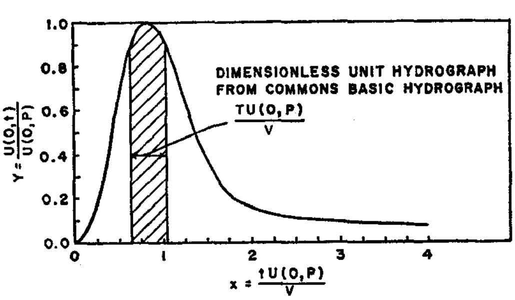

FIG. 5 – Dimensionless unit hydrograph, from Commons basic hydrograph

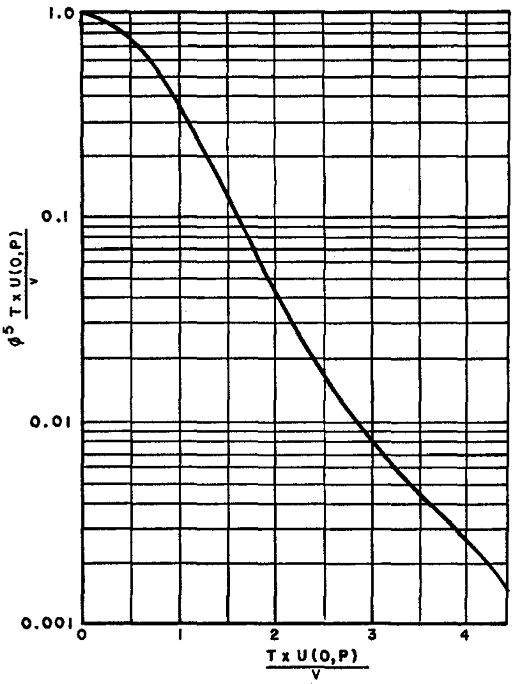

FIG. 6 – Evaluation of ϕ^5 $[TU(0, P)/V]$

instantaneous unit hydrographs were all reasonably close to the dimensionless unit hydrograph derived from *Commons* [1942] basic hydrograph. Accordingly, (9) can be used to obtain the peak of the instantaneous unit hydrograph and the standard dimensionless unit hydrograph, Figure 5, may be used to complete the instantaneous unit hydrograph.

Eq. (12) can now be simplified as follows. In Figure 5 the shaded area is bounded by equal ordinates separated by $TU(0, P)/V$. Therefore the shaded area is the maximum area between any two ordinates separated by this amount. The mean ordinate of this area is a function of $TU(0, P)/V$, say

$$\varphi(TU(0, P)/V) = \frac{\int y\, dx}{TU(0, P)/V}$$

between appropriate limits

$$= \frac{\int \dfrac{U(0, t)}{U(0, P)} \times \dfrac{U(0, P)}{V}\, dt}{TU(0, P)/V}$$

the integration to be taken between two ordinates separated by T and chosen in position so as to obtain the maximum value of the integral.

$\therefore \varphi(TU(0,P)/V)$

$$= \frac{1}{U(0,P)} \int \frac{U(0,t)}{T} dt = \frac{U(T,P)}{U(0,P)}$$

or

$$U(T,P) = U(0,P)\varphi[TU(0,P)/V] \qquad (13)$$

$\varphi(TU(0,P)/V)$ is very easily calculated from the dimensionless unit hydrograph. The fifth power of this function is shown in Figure 6.

We have now expressions for $F(R, T$ or less, $0)$, $U(T, P)$ and $U(0, P)$. These may be inserted in (7) to obtain the particular solution for well-drained catchments in Ireland.

TABLE 1—*Evaluation of Eq. 15 for U(O, P) = 100 ft³/sec*

					$U(O,P) = 100$											
n	$\frac{50n}{V}$	ϕ^5	$n\phi^5$	$T = \frac{50n}{U(O,P)}$	A = 5 sq mi		A = 20 sq mi		A = 50 sq mi		A = 100 sq mi		A = 200 sq mi		A = 500 sq mi	
					u	$n\phi^5u$	u	$n\phi^5u$	u	$n\phi^5u$	u	$n\phi^5u$	u	$n\phi^5u$	u	$n\phi^5u$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	.155	.97	1.94	1	.68	1.319	.20	.388	.10	.194	.045	.087	.013	.025	.001	.002
4	.310	.90	3.60	2	.69	2.484	.205	.738	.107	.385	.045	.162	.014	.050	.0015	.005
6	.465	.79	4.74	3	.695	3.294	.22	1.043	.115	.545	.045	.213	.015	.071	.002	.009
8	.620	.66	5.28	4	.71	3.749	.25	1.320	.120	.634	.050	.264	.019	.100	.003	.016
10	.775	.53	5.30	5	.72	3.816	.28	1.484	.137	.726	.062	.329	.025	.132	.0045	.024
12	.930	.40	4.80	6	.73	3.504	.305	1.464	.155	.744	.075	.360	.030	.144	.006	.029
14	1.085	.30	4.20	7	.74	3.108	.34	1.428	.175	.735	.085	.357	.035	.147	.008	.034
16	1.240	.218	3.49	8	.75	2.618	.37	1.292	.200	.698	.100	.349	.045	.157	.011	.038
18	1.395	.155	2.79	9	.76	2.120	.40	1.116	.215	.600	.115	.321	.050	.140	.0135	.038
20	1.550	.110	2.20	10	.77	1.694	.42	.924	.23	.506	.125	.275	.060	.132	.017	.037
22	1.705	.079	1.74	11	.775	1.349	.445	.774	.25	.435	.140	.244	.065	.113	.020	.035
24	1.860	.057	1.37	12	.785	1.075	.465	.637	.265	.363	.155	.212	.075	.103	.025	.034
26	2.015	.042	1.09	13	.795	.867	.485	.529	.28	.305	.17	.185	.085	.093	.030	.033
28	2.170	.0315	.882	14	.805	.710	.505	.445	.300	.265	.185	.163	.095	.084	.033	.029
30	2.325	.0235	.705	15	.815	.575	.53	.374	.31	.219	.20	.141	.10	.071	.038	.027
32	2.480	.0177	.566	16	.820	.464	.55	.311	.33	.187	.21	.119	.11	.062	.042	.024
34	2.636	.0135	.459	17	.830	.381	.56	.257	.345	.158	.22	.101	.12	.055	.047	.022
36	2.791	.0107	.385	18	.835	.321	.58	.223	.355	.137	.23	.088	.13	.050	.052	.020
38	2.946	.0086	.327	19	.840	.275	.59	.193	.37	.121	.25	.082	.135	.044	.057	.019
40	3.101	.0070	.280	20	.845	.237	.60	.168	.38	.106	.26	.073	.14	.039	.060	.017
42	3.256	.0059	.248	21	.850	.211	.61	.151	.39	.097	.27	.067	.15	.037	.065	.016
44	3.411	.0050	.220	22	.855	.188	.62	.136	.405	.089	.28	.060	.16	.035	.070	.015
46	3.566	.00425	.196	23	.860	.169	.63	.123	.42	.082	.29	.056	.165	.032	.075	.015
48	3.721	.0036	.173	24	.862	.149	.645	.112	.43	.074	.30	.051	.175	.030	.080	.014
50	3.876	.0030	.150	25	.867	.130	.655	.098	.445	.067	.31	.046	.185	.028	.085	.013
52	4.031	.0026	.135	26	.871	.118	.665	.090	.46	.062	.32	.043	.195	.026	.090	.012
54	4.186	.0022	.119	27	.875	.104	.675	.080	.47	.056	.33	.039	.205	.024	.095	.011
56	4.341	.0019	.106	28	.880	.093	.685	.073	.485	.051	.34	.036	.21	.022	.100	.011
58	4.496	.0016	.093	29	.882	.082	.69	.064	.495	.046	.35	.032	.22	.020	.110	.010
60	4.651	.0014	.084	30	.885	.074	.70	.059	.51	.043	.36	.030	.23	.019	.115	.010
Sum from n = 2 to n = 60					...	35.278	...	16.094	...	8.730	...	4.585	...	2.085	...	.619
Sum from n = 62 to $n = \infty$					...	.637	...	.558	...	.445	...	.361	...	.269	...	.184
Sum from n = 2 to $n = \infty$					...	35.915		16.652		9.175		4.946		2.354		.803
$4\Sigma n\mu\phi^5$					...	143.66	...	66.608	...	36.700	...	19.784	...	9.416	...	3.212
$\mu(O, A)$					...	.68	...	.20	...	.10	...	.045	...	.013	...	.002
$\mu(O, A) + 4\Sigma n\mu\phi^5$					...	144.34	...	66.808	...	36.800	...	19.829	...	9.429	...	3.214
$\frac{QF_Q^{0.2}}{P} \times \frac{40}{M}$					...	526	...	1800	...	4000	...	7080	...	12200	...	24.600
Catchment slope					...	260	...	450	...	650	...	850	...	1100	...	1600

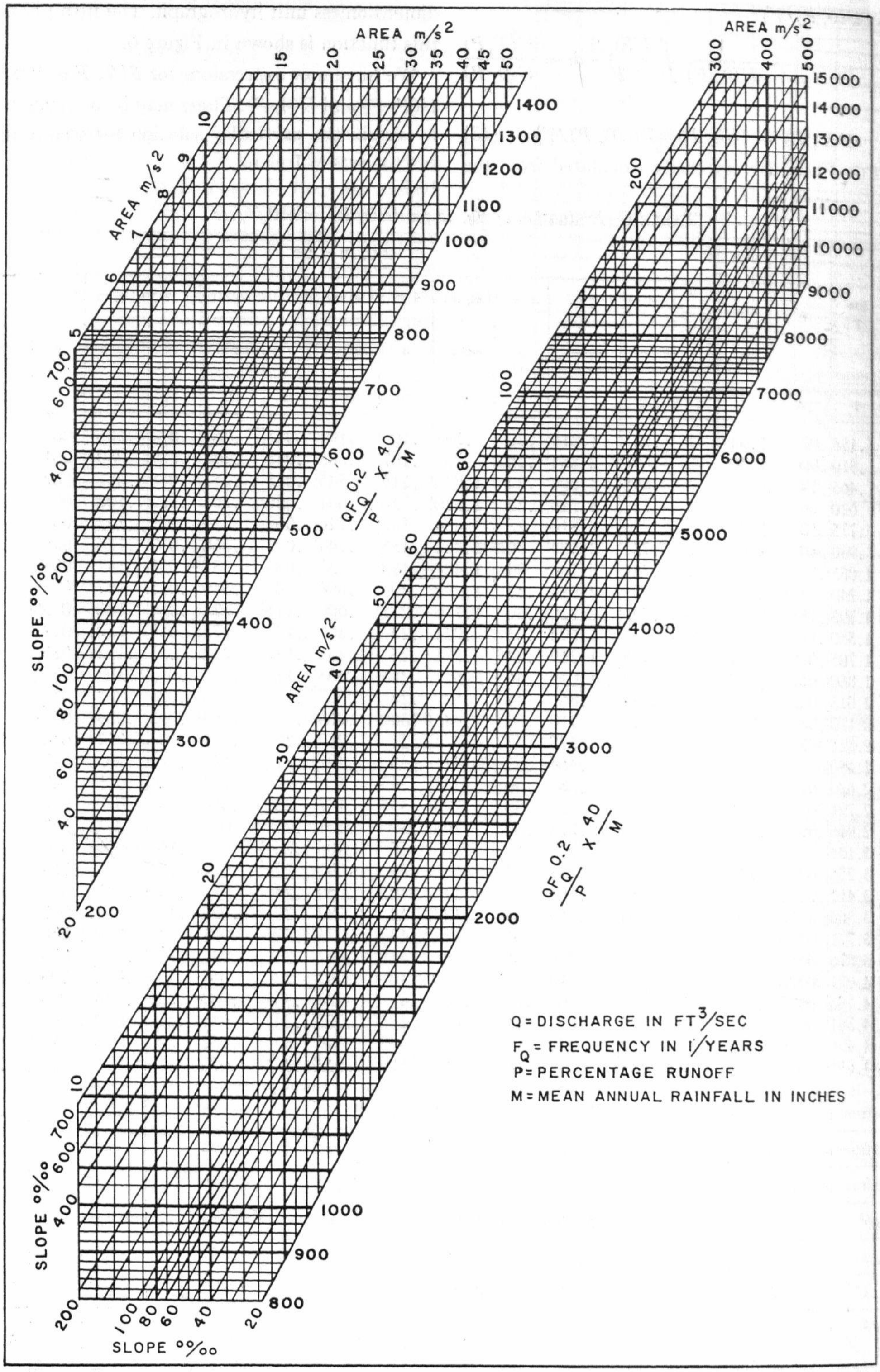

FIG. 7 – The frequencies of discharges from well drained catchments in Ireland

$$F_q = \mu(0, A)(0.036)\left(\frac{M}{40}\right)^5 h^2 \frac{P^5U^5(0, P)}{q^5}$$

$$+\Sigma\mu(T, A)\left[0.036\left(\frac{M}{40}\right)^5 (T+h)^2 \frac{P^5U^5(T, P)}{q^5} - 0.036\left(\frac{M}{40}\right)^5 (T-h)^2 \frac{P^5U^5(T, P)}{q^5}\right]$$

for $T = 2h, 4h, 6h$, etc.

$$= 0.036 \frac{P^5U^5(0, P)}{q^5}\left(\frac{M}{40}\right)^5 \cdot [h^2\mu(0, A) + \Sigma\mu(T, A)\, 4Th\varphi^5(TU(0, P)/V)] \quad (14)$$

for $T = 2h, 4h, 6h$, etc. As h is arbitrary, provided it is small, we may take $h = 50$ ft^3 hr sec^{-1}/$U(0, P)$ (taking h as being inversely proportional to $U(0, P)$ simplifies the subsequent calculations). We shall also define n by the equation $T = nh$. Hence $T = 50n/U(0, P)$ and $TU(0, P)/V = 50n/V$

Substituting for T and h in (14)

$$F_q = 0.036 \frac{P^5U^5(0, P)}{q^5}\left(\frac{M}{40}\right)^5 \left\{\mu(0, A)\left[\frac{50}{U(0, P)}\right]^2 + 4\Sigma\mu(T, A)n \left[\frac{50}{U(0, P)}\right]^2 \varphi^5\left(\frac{50n}{V}\right)\right\}$$

for $h = 2, 4, 6$ etc., or

$$\frac{Q^5F_Q}{P^5} \times \left(\frac{40}{M}\right)^5 = 0.036A^5(50)^2U^3(0, P) \left[\mu(0, A) + 4\Sigma\mu(T, A)n\varphi^5\left(\frac{50n}{V}\right)\right] \quad (15)$$

for $n = 2, 4, 6$ etc. The right hand side of this equation is a function of catchment slope and area. Table 1 shows the evaluation of this factor for catchments of various areas having $U(0, P) = 100$ ft^3/sec. Column 3 is obtained from Figure 6. Column 6 is obtained from Figure 2; columns 7, 9, 11, 13, 15 and 17 give the values of the elements within the Σ sign in (15) for $n = 2, 4, 6, 8$, etc. These columns are totaled and an approximation made to the sum of the remaining terms to infinity by assuming that the remaining terms form a geometric progression whose common ratio is the fourth root of the ratio of the 60th term to the 56th term. The catchment slope corresponding to $U(0, P) = 100$ ft^3/sec and $A = 5, 20, 50, 100, 200,$ and 500 sq mi respectively is obtained from (9). Similar calculations were made for other values of $U(0, P)$ and when the final result was plotted on logarithmic paper, it was found that the data could be represented by a series of straight lines, one for each value of S, Figure 7. The portion of the diagram in the top left hand corner is an extension of the main portion downwards and to the left.

In this chart, as in the general theory, no allowance has been made for ground water. Where this is significant, an addition of one to two cubic feet per second per square mile should be made.

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Engineer's Branch, Office of Public Works, Dublin, Ireland

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