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Correlogram Analysis Revisited

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Abstract. Consideration of the Hurst coefficient h offers new insight into the interpretation of observed correlograms for streamflow sequences. Autoregressive models, for which $h = \frac{1}{2}$, cannot reproduce the structures of those correlograms. The structures are symptomatic of long-term persistence as indicated by observed values of h being greater than $\frac{1}{2}$. The tendency has been to test a sequence for independence, and if the hypothesis of independence is unacceptable, then (1) the generating process for the sequence is approximated by a short memory process, and (2) variations in the high lag serial correlation coefficients are ascribed to chance. If indeed the sequences are generated by long memory processes, then the powers of most independence tests are low for short records. Consequently, correlograms may be useful indicators of long-term persistence when more formal tests give contrary results.

The tendency for high flows to follow high flows and for low flows to follow low flows is referred to as hydrologic persistence and is attributed to storage processes in the atmosphere or in the drainage basin, either surface or subsurface. This tendency is marked in some streamflow sequences and obscured or absent in others. The detection and quantitative measurement of persistence is important in the design of long-term storage reservoirs. To produce the same estimate of firm yield larger reservoirs are needed as persistence in the record increases. In this paper the nature of persistence, its definition, and the means for its detection are discussed. Simulation is used to estimate our ability to detect its presence or absence. The simulations make extensive use of two types of stochastic processes, first, a classic short memory process, the lag-one Markov; and second, the type 2 discrete approximation to fractional noise, which we shall regard as the paradigm of long memory processes.

Persistence is described by the structure of serial dependence of a streamflow sequence. A quantitative measure of this structure is given by the serial correlation function, the correlogram (ρ_u versus u), where ρ_u is the serial correlation coefficient for lag u. An observed correlogram is subject to sampling errors, particularly for the estimated values of ρ_u where u is large, and it is difficult to estimate the population correlogram that pertains to the sequence generating process from a small finite sample.

Another measure of persistence is provided by the Hurst coefficient h, which is defined as

$$R/S \sim N^h \tag{1}$$

where R is the range of cumulative departures from the sample mean, S is the sample standard deviation, and N is the sample size, sequence length. For purely random normal processes with N large, $h = \frac{1}{2}$ [*Hurst*, 1951; *Feller*, 1951]. Hurst investigated many natural phenomena and found the average value of estimates of h to be 0.73 with a standard deviation of 0.08. The tendency for natural sequences to yield estimates of $h > \frac{1}{2}$ is referred to as the Hurst phenomenon.

All stochastic processes belonging to the Brownian domain of attraction are characterized by $h = \frac{1}{2}$. Purely random and autoregressive processes belong to this domain. For these processes, h provides no measure of persistence. Quite often, the lag-one serial correlation is taken to be a measure of persistence. This is appropriate if indeed a sequence is generated by a first order autoregressive process (Markov process), because for this process $\rho_u = \rho_1^{|u|}$. For a vth order autoregressive process, the correlogram and persistence are defined by $\rho_1, \cdots, \rho_v$. Fractional noise processes lie outside the Brownian domain and may assume values of h in the range (0, 1) with the exception of $h = \frac{1}{2}$. For fractional noise processes, ρ_u and h are related measures of persistence.

Correlograms derived from annual streamflow sequences are characterized by the large variability of estimates of ρ_u. The tendency has been to write off this variability as a time sampling error, particularly for u large, and to approximate the generating process of the sequences by a low order autoregressive process, generally first order, for those streams where estimates of the low order serial correlation coefficients are statistically significant. By writing off the variablity of the estimates of ρ_u for u large as a time sampling error, then only short-term persistence at best can be inferred from the correlogram.

This paper revisits correlogram analysis in light of Markovian and fractional noise generating processes. Real world correlograms when compared to those generated by these processes are indicative of long-term persistence, yet various tests for serial dependence fail to reflect such persistence because of the large type 2 errors associated with magnitudes of N and ρ encountered in hydrology.

GENERATING PROCESSES

A Markov process is defined as

$$X(t) = \rho X(t-1) + \epsilon(t) \tag{2}$$

where $X(t)$ and $X(t-1)$ denote events at time t and $t-1$, respectively, and $\epsilon(t)$ is a random component, independent of $X(t-1)$. For this process

$$\rho_u = \rho^{|u|} \tag{3}$$

where $\rho_1 \equiv \rho$, and $h = \frac{1}{2} \; \forall \rho$.

Fractional noise processes are continuous parameter processes that have infinite memories, and so, to be operational, they must be approximated as discrete parameter processes with large but finite memories. One approximation, referred to as type 2 [*Mandelbrot and Wallis*, 1969], is given by

$$X(t) = (h - \tfrac{1}{2}) \sum_{u=t-M}^{t-1} (t-u)^{h-3/2} \epsilon(u) \tag{4}$$

where $X(t)$ is the event at time t and $\epsilon(u)$ is a random component. The Hurst coefficient is h and the length of the finite memory is M. For the type 2 process it can be shown that ρ_u is given by

$$\rho_u = \sum_{i=0}^{M-1-u} [(M-i)(M-i-u)]^{h-3/2} \div \sum_{i=0}^{M-1} (M-i)^{2h-3} \tag{5}$$

The type 2 process is characterized by low frequency components, such that for large values of M, say $M > 1000$, $\rho_1 > 0.6$; therefore the type 2 process can approximate the generating process for but a few annual streamflow sequences. To reduce the dominance of the low frequency components and thereby to allow ρ_1 to assume values nearer zero, *Matalas and Wallis* [1971] considered a filtered type 2 process defined as

$$Y(t) = (h - \tfrac{1}{2}) \sum_{u=pt-M}^{pt-1} (pt-u)^{h-3/2} \epsilon(u) \tag{6}$$

where $p \geq 1$ is an integer and $Y(t) = X(pt)$ is the event generated by (4) at time pt. For all values of p, the mean, variance, and skewness for $X(t)$ and $Y(t)$ are the same, but

$$\rho_u(y) = \rho_{pu}(x) \tag{7}$$

From (5) and (7), h is seen to be directly related to ρ_u.

Without loss of generality, assume $E[X(t)] = 0$ and $E[X(t)]^2 = 1$. Then for the Markov process

$$E[X(t)X(t-k) \mid X(t-1), \cdots, \cdot X(t-k+1)] = 0 \; \forall \; k \geq 2 \tag{8}$$

This property is referred to as the short memory or short-term persistence of the Markov process. For the filtered type 2 process

$$E[X(t)X(t-k) \mid X(t-1), \cdots, \cdot X(t-k+1)] = 0 \; \forall \; k > M \tag{9}$$

For M large, the process is characterized by long memory or long-term persistence.

Sequences of length $N = 1000$ generated by (2) and (6) are shown in Figures 1 and 2. Figure 1 shows samples of filtered type 2 processes for various values of h and p with $M/p = 1000$. Figure 2 shows samples of Markov processes for various values of ρ that correspond to the values of ρ for the filtered type 2 processes.

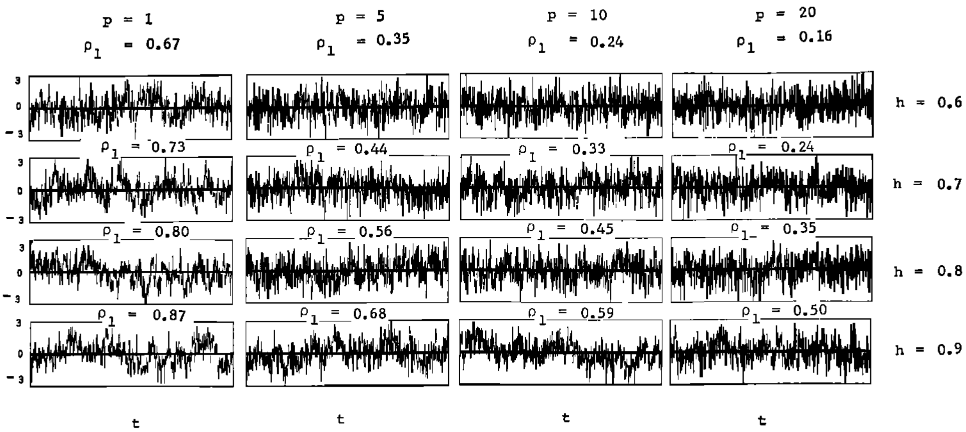

Fig. 1. Sample functions ($t = 1, \cdots, 1000$) of filtered type 2 fractional Gaussian noises of varying h and p, with $M/p = 1000$.

From Figure 1, it is seen that as h increases for p fixed, the low frequency components become a dominant feature of the sequences. For h fixed this dominance decreases as p increases. For the Markov sequences (Figure 2), the intermediate frequency components become more prevalent as ρ increases. The extreme low frequency components are absent from the Markov sample functions, even for high ρ. The apparent lack of high frequency components in the high ρ Markov sample functions may be partially explained by the absence of the small sample variance correction in the generating mechanism [*Matalas*, 1967]. The Markov and filtered type 2 processes yield sequences that are not easily distinguished in those cases where ρ is the same for the two processes, particularly when N or ρ is small.

When extreme low frequency components are present in a stochastic process, sample functions may have the appearance of 'hidden periodicities.' In the absence of information about the generating process, an examination of sample functions, particularly those where h is large, could easily lead to an infection of 'cycle fever.' The statistics of cycle detection when $h \neq 0.5$ are yet to be evaluated.

ESTIMATION OF h

The estimate of h, denoted by K, proposed by *Hurst* [1951] is given as

$$K = \log (R/S)/\log (N/2) \tag{10}$$

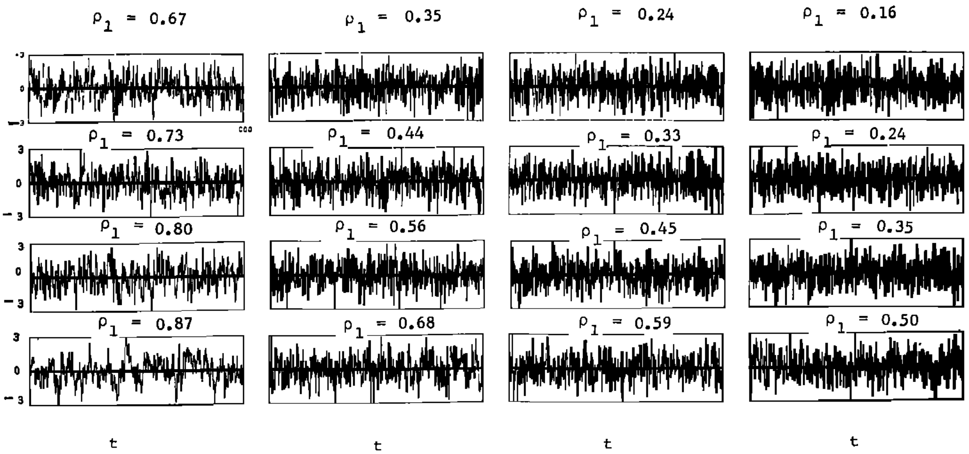

Fig. 2. Sample functions ($t = 1, \cdots, 1000$) of Gaussian lag-one Markov noises.

TABLE 1. Mean of 10,000 Estimates of K for Markovian Processes

	ρ									
N	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
5	0.71	0.72	0.73	0.74	0.75	0.75	0.76	0.77	0.77	0.78
10	0.69	0.70	0.72	0.73	0.76	0.77	0.79	0.80	0.82	0.83
25	0.65	0.68	0.71	0.73	0.75	0.77	0.80	0.82	0.85	0.87
50	0.63	0.66	0.68	0.71	0.73	0.75	0.77	0.80	0.84	0.87
100	0.61	0.65	0.66	0.69	0.71	0.73	0.75	0.79	0.82	0.87

Mandelbrot and Wallis [1969] estimated h in the following manner. A time series of N observations is divided into a set of time series, referred to as subseries, of length n, where $5 \leq n \leq N$. For each subseries of length n, $R(n)$ and $S(n)$ are determined. The notation $R(n)$ and $S(n)$ is used to indicate that these statistics are derived from a sequence of length n. In equation 10, R and S may be denoted as $R(N)$ and $S(N)$. The slope of log $[R(n)/S(n)]$ versus log n is denoted as H and is the estimate of h.

Wallis and Matalas [1970] calculated H by the method of least squares and studied some of the distributional properties of K and H for various values of N. These studies showed that both K and H overestimate h for purely random and Markovian processes. The bias for K was larger than that for H, but the variance of K was smaller than that for H. For Markovian processes the biases for K and H increase as ρ increases. Fractional noise processes also yield biased estimates of h. For $h > 0.7$, K and H underestimate h, and for $h > 0.7$, K and H overestimate h. In all cases, the biases and variances were found to decrease slowly with an increase in N.

In Table 1, the mean values of K are given for various values of N and ρ for Markovian processes. The case of a purely random process is given by $\rho = 0$. For filtered type 2 processes, the mean values of K for various values of N, h, and p are given in Table 2.

Matalas and Huzzen [1967] presented a table of mean values of K for various values of N and ρ for Markovian processes. These values differ somewhat from those presented in Table 1 for $N = 5$, 10, 25 because of the manner used in calculating the standard deviation S for the various sequences. Matalas and Huzzen obtained S by extracting the square root of the sum of squared deviations from the sequence mean divided by $N - 1$. The values in Table 1 are based on the use of N instead of $N - 1$ as the divisor in calculating S. J. S. Smart (personal communication, 1971) has pointed out that the use of $N - 1$ does not lead to unbiased estimates of σ and it is not known if R is a biased estimator, so perhaps S should be based on the use of N.

The mean values of K in Table 1 are obtained from the Matalas-Huzzen values K_{MH} by

$$K = \frac{1}{2} \frac{\log [N/(N-1)]}{\log (N/2)} + K_{MH} \qquad (11)$$

where K tends to K_{MH} as N increases. For $N > 25$, the difference between K and K_{MH} is negligible.

Estimates of H comparable to those given for K (Table 1) for the Markovian world can be extracted from the figures presented in *Wallis and Matalas* [1970]. Estimates of H for some

TABLE 2. Mean Values of K for Filtered Type 2 Processes Based on Series of Length 50,000

	$P = 1$				$P = 5$				$P = 10$				$P = 20$			
N	$h =$ 0.6	$h =$ 0.7	$h =$ 0.8	$h =$ 0.9	$h =$ 0.6	$h =$ 0.7	$h =$ 0.8	$h =$ 0.9	$h =$ 0.6	$h =$ 0 7	$h =$ 0.8	$h =$ 0 9	$h =$ 0.6	$h =$ 0.7	$h =$ 0.8	$h =$ 0.9
25	0.79	0.81	0 81	0.82	0.73	0.75	0.76	0.78	0.70	0.72	0.74	0.76	0.69	0.70	0.72	0.74
50	0.80	0.81	0.83	0.84	0.73	0.75	0.77	0.79	0.70	0.72	0.75	0.78	0.68	0.71	0.73	0.76
100	0 81	0.82	0.83	0.85	0.74	0.76	0.78	0.81	0.71	0.73	0.75	0.79	0.68	0.71	0.74	0.78

TABLE 3. Mean Values of H for Filtered Type 2 Fractional Noises and the Hurst Procedure Based on Series of Length 50,000

	$P = 1$				$P = 5$				$P = 10$				$P = 20$			
N	$h =$ 0.6	$h =$ 0.7	$h =$ 0.8	$h =$ 0.9	$h =$ 0.6	$h =$ 0.7	$h =$ 0.8	$h =$ 0.9	$h =$ 0.6	$h =$ 0.7	$h =$ 0.8	$h =$ 0.9	$h =$ 0.6	$h =$ 0.7	$h =$ 0.8	$h =$ 0.9
25	0.82	0.83	0.84	0.86	0.72	0.75	0.79	0.81	0.68	0.71	0.75	0.79	0.66	0.68	0.73	0.75
50	0.82	0.84	0.87	0.88	0.74	0.77	0.80	0.82	0.69	0.72	0.76	0.81	0.65	0.70	0.74	0.78
100	0.83	0.84	0.86	0.89	0.74	0.77	0.80	0.83	0.69	0.73	0.76	0.82	0.66	0.70	0.75	0.80

typical type 2 filtered fractional noises are given in Table 3.

The values shown in Table 1 are conditional expectations of K for given values of ρ. For a given sequence estimates of h, given by K, and of ρ, given by $\hat{\rho}$, are correlated. The correlation, which is a function of N and ρ, increases as ρ increases and decreases as N increases. For values of $N = 50$ and 100, the correlation between K and $\hat{\rho}$ for various values of ρ is shown in Figure 3.

CORRELOGRAMS

Empirical correlograms graphically display the relation $\hat{\rho}_u$ versus u. Because of the variability in the estimates $\hat{\rho}_u$, particularly for u large, it is difficult to infer the generating process giving rise to the correlogram. For the most part, values of $\hat{\rho}_u$ for large values of u are assumed to be subject to such large errors as to be of little value in the interpretation of correlograms. A brief look at some correlograms for real world and derived sequences indicates that despite the variability of $\hat{\rho}_u$, the correlograms do reflect the presence of short-term and long-term persistence.

In Figure 4, the correlogram is shown for annual adjusted virgin flows of the Colorado River at Lees Ferry, Arizona [*Leopold,* 1959].

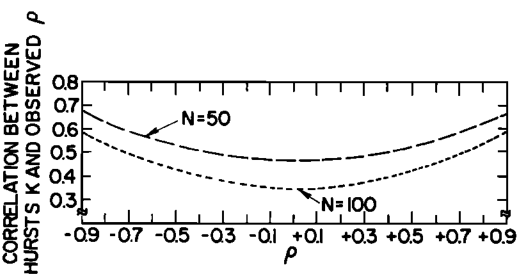

Fig. 3. Correlation between estimates of K and P_1 for Markov processes.

This sequence of length $N = 61$ yields $\hat{\rho}_1 = 0.21$ and $K = 0.82$. Based on a Markov process with $\rho_1 = 0.22$, sequences of length $N = 100$ were generated. Among these sequences, those yielding a value of $\hat{\rho}_1 = 0.21$ were selected, one of which is shown in Figure 5. To generate Markov sequences of length $N = 100$ with an expected value of $K = 0.82$, ρ_1 was taken to be 0.8 (Table 1). Among the generated sequences, those yielding a value of $K = 0.82$ were selected, one of which is shown in Figure 6. As we show later, it would be improbable to find Markovian sequences of length 100 that preserve both the $\hat{\rho}$ and K of the Colorado River data.

For various values of h and p with $M = 1000$, filtered type 2 sequences of length $N = 100$ were generated. Among these sequences, those yielding values of $\hat{\rho}_1 = 0.22$ and $K = 0.82$ were selected. Correlograms for two such sequences are shown in Figures 7 and 8. In Figure 8, the underlying correlogram ρ_u versus u is shown.

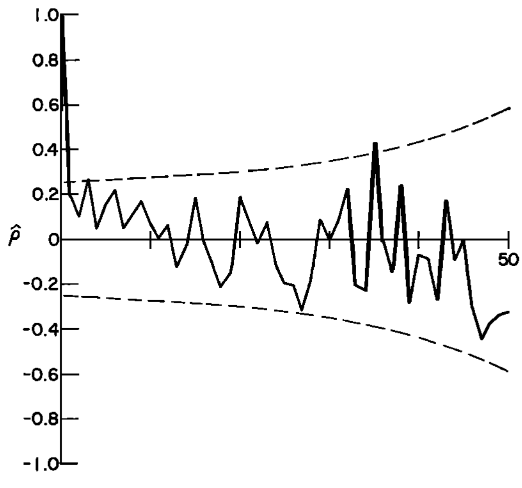

Fig. 4. Correlogram, Colorado River at Lees Ferry, Arizona, $N = 61$, $K = 0.82$, $\hat{\rho} = 0.21$.

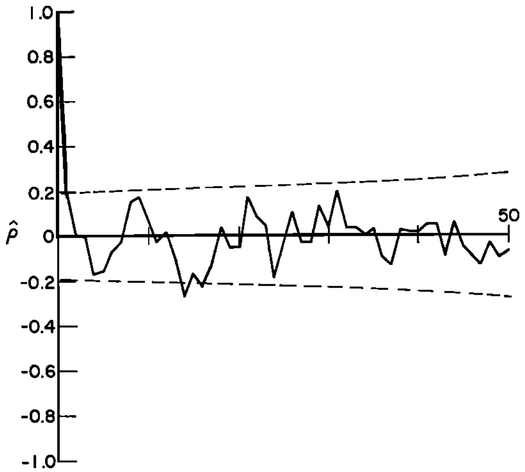

Fig. 5. Correlogram, sample of Markov process, $N = 100$, $\rho = 0.21$, $\hat{\rho} = 0.21$.

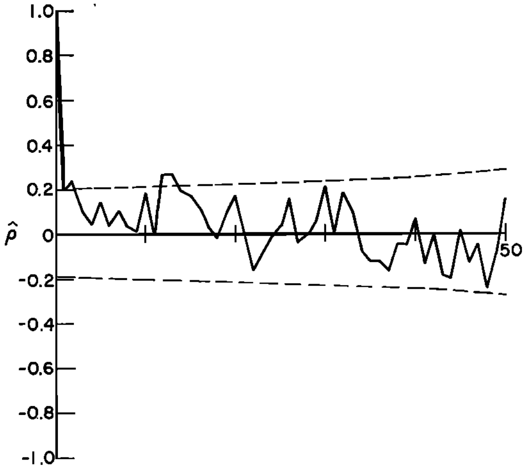

Fig. 7. Correlogram, sample of filtered type 2 process, $N = 100$, $h = 0.6$, $p = 20$, $M = 1000$, $\hat{\rho} = 22$, $K = 0.82$.

TESTS FOR INDEPENDENCE

Given a sequence of N events $X(1), \cdots, X(N)$, estimates of ρ_u, denoted by $\hat{\rho}_u$, are obtained in the following manner. For a given value of $u \geq 0$, two sequences are formed from the given sequence $X(1), \cdots, X(N-u)$ and $X(u+1), \cdots, X(N)$. If the two sequences are denoted as $X_1(u)$ and $X_2(u)$ and are considered as concurrent, then

$$\hat{\rho}_u = C[X_1(u)X_2(u)]/SD[X_1(u)]SD[X_2(u)] \tag{12}$$

where $C[X_1(u)X_2(u)]$ is the covariance of $X_1(u)$ and $X_2(u)$ and $SD[X_1(u)]$ and $SD[X_2(u)]$ are the standard deviations of $X_1(u)$ and $X_2(u)$, respectively.

The values of ρ_u versus u, the correlogram, reflect the structure of the time dependence for a sequence. To facilitate the analysis of this structure the correlogram is depicted graphically with ρ_u the ordinate and u the abscissa with adjacent points connected by straight lines.

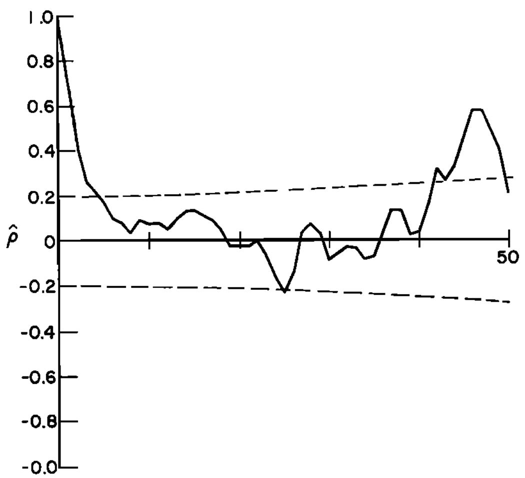

Fig. 6. Correlogram, sample of Markov process, $N = 100$, $\rho = 0.80$, $K = 0.82$.

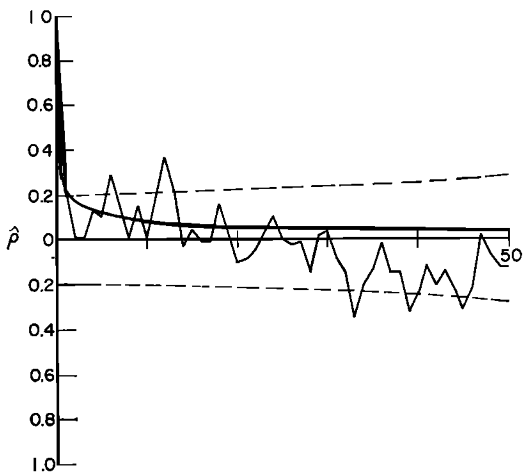

Fig. 8. Correlogram, sample of filtered type 2 process, $N = 100$, $h = 0.7$, $p = 20$, $M = 1000$, $\hat{\rho} = 0.21$, $K = 0.82$.

For an observed sequence, the empirical correlogram $r(u)$ versus u must be used to infer the underlying correlogram ρ_u versus u. Because N is finite, the values of $\rho(u)$ are subject to sampling errors that increase with u and with $1/N$.

An often noted property of hydrologic correlograms is their tendency to produce significant correlations at large u. The search for cyclic explanations of the occasional large ρ_u has been hampered by the instability of the observed wavelengths among different records, the lack of physical models, and the unknown power of statistical cycle tests in the presence of long-term dependence. It is our belief that filtered type 2 fractional noise correlograms and hydrologic records of comparable N, ρ, and K are inseparable from each other. To paraphrase this last remark, we believe that many hydrologic records do exhibit long-term persistence, that their correlograms show this in a subjective manner. We shall now attempt to show why we believe that objective statistical tests have failed to confirm the visual evidence of Figures 4–8.

To test an observed sequence for serial independence involves testing the hypothesis that a sequence is generated by a purely random process $\rho_u \neq 0 \ \forall \ u > 0$ against the alternate hypothesis that the sequence is generated by a process for which $\rho_u \neq 0 \ \forall \ u > 0$. In hydrologic studies, one of the most commonly used tests is the *Anderson* [1942] test. To a lesser extent various runs tests have been used. The test procedures are based on the assumption that the sequences are stationary.

Anderson's test assumes a normal process and a circular sequence. Circularity means that the sequence closes on itself; that is, $X(1)$ follows $X(N)$. For such a sequence, $\hat{\rho}(u)$ is defined as follows:

$$\hat{\rho}(u) = C[X_1(u)X_2(u)]/V(x) \qquad (13)$$

where $X_1(u)$ and $X_2(u)$ are the two sequences formed from $X(1), \cdots, X(N)$ as noted above and $V(x)$ is the variance of $X(1), \cdots, X(n)$. Values of $\hat{\rho}(u)$ determined from equations 12 and 13 are not very different if u is small relative to N.

$\rho(1)$, determined by (13), is approximately normally distributed with mean $-1/(N-1)$ and variance $(N-2)/(N-1)^2$ with the confidence limits for $\hat{\rho}(1)$ given by

$$\text{C.L. } [\hat{\rho}(1)] = [-1 \pm Z_\alpha(N-2)^{1/2}]/(N-1) \qquad (14)$$

where Z_α is the normal deviate corresponding to probability α. If $\hat{\rho}(1)$ falls outside the confidence limits, then the hypothesis that $\rho(1) = 0$ is not accepted.

Based on the theory of runs, various tests for serial independence may be devised. One such test, discussed by *Fisz* [1963], is as follows. Given a sequence $X(1), \cdots, X(N)$, the median $\tilde{X}$ is determined and a sequence is formed by replacing $X(i)$ with 0 if $X(i) < \tilde{X}$ and with 1 if $X(i) \geq \tilde{X}$. If the original sequence has been generated by a purely random process, then n, the number of times 0 is followed by 1 or 1 is followed by 0, is approximately normally distributed with mean $(N-1)/2$ and variance $(N-1)/4$, whereby, the confidence limits for n are given by

$$\text{C.L. } (n) = [(N-1) \pm Z_\alpha(N-1)^{1/2}/2] \qquad (15)$$

If n lies within the limits, then the hypothesis that the sequence is generated by a purely random process is accepted.

Another test procedure, described by *Kendall* [1951], is to form a sequence from $X(1), \cdots, X(N)$ in the following manner. If $X(i) > X(i-1)$ and $X(i) > X(i+1)$ or if $X(i) < X(i-1)$ and $X(i) > X(i+1)$, then $X(i)$ is assigned the value 0; otherwise, $X(i)$ is assigned the value 1. The number of 1, n', is approximately normally distributed with mean $2(N-1)/3$ and variance $(16N-29)/30$, whereby, the confidence limits for n are given by

$$\text{C.L. } (n') = \{2(N-2)/3 \pm Z_\alpha[(16N-29)/90]^{1/2}\} \qquad (16)$$

The hypothesis that the sequence is generated by a purely random process is accepted if n' lies within the confidence limits.

Still another test has been suggested by *Gold* [1929]. For the sequence $X(1), \cdots, X(N)$, a run of length m is defined by a set of m consecutive events either above or below the median. If $n''(m)$ denotes the total number of runs above and below the median of length m, then for a purely random process, the expected value of $n''(m)$ is given by

$$E[n''(m)] = [N + (3-m)]/(2^{m+1}) \qquad (17)$$

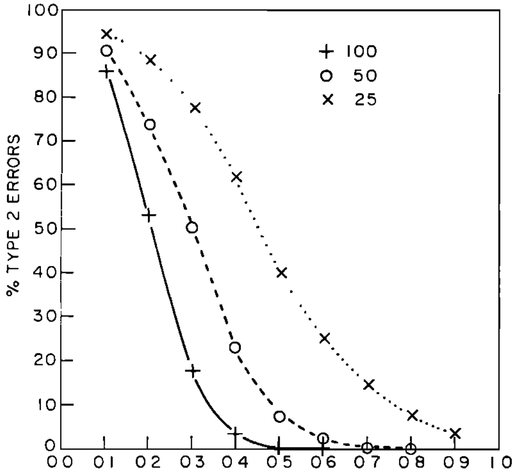

Fig. 9. Power of Anderson test, Markov processes.

where $\sum_{m=1}^{m'} \{n''(m) - E[n''(m)]\}^2/E[n''(m)]$ is distributed as chi square with $m' - 1$ degrees of freedom, where m' is the maximum run length in the sequence.

The three runs tests described above are referred to hereafter as test 1, test 2, and test 3, the order being that in which they have been presented.

The four tests described above do not constitute the universe of tests for independence. The Anderson test is widely used in hydrologic studies whereas runs tests have been used to a lesser extent. In applying tests of independence, little attention has been paid to their power. In testing a hypothesis, such as whether a sequence is generated by a purely random process, two types of errors must be considered. The type 1 error is the probability of rejecting the hypothesis when it is true, and the type 2 error is the probability of accepting the hypothesis when it is false. In general the size of the type 2 error will increase as the type 1 error decreases. The power of a test is defined as $1 - \beta$, where β is the type 2 error. The type 1 error may be denoted as α.

Given $\alpha = 0.05$, which is usually the case in hydrologic studies, β is evaluated for each of the above described tests for various values of N based on a large number of sequences generated by Markovian and filtered type 2 processes.

Figures 9 and 10 show β versus ρ for the Anderson test for $N = 25$, 50, and 100 for Markovian and filtered type 2 processes. For both types of processes β decreases as ρ increases for

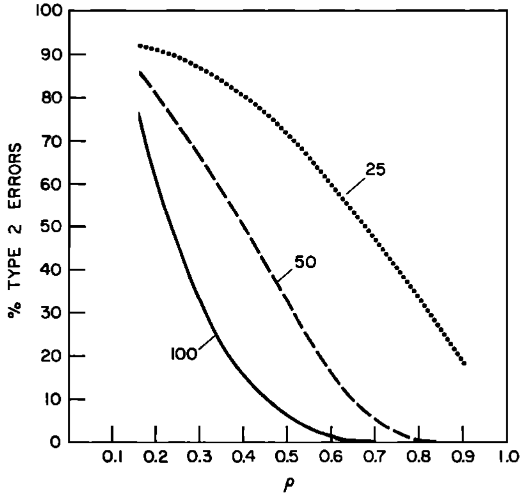

Fig. 10. Power of Anderson test, filtered type 2 processes.

a fixed value of N, and β decreases as N increases for a fixed value of ρ. The values of β for a given value of ρ are not sensitive to values of $0.6 \leq h \leq 0.9$ for filtered type 2 processes. For given values of N and ρ, β is smaller for Markov processes than for filtered type 2 processes. If $\rho = 0.3$ is taken as the average value of lag-one serial correlation with $N = 50$ being the average value of observed streamflow sequences, then $\beta \approx 65\%$ for both Markovian and filtered type 2 processes. Under these conditions, the Anderson

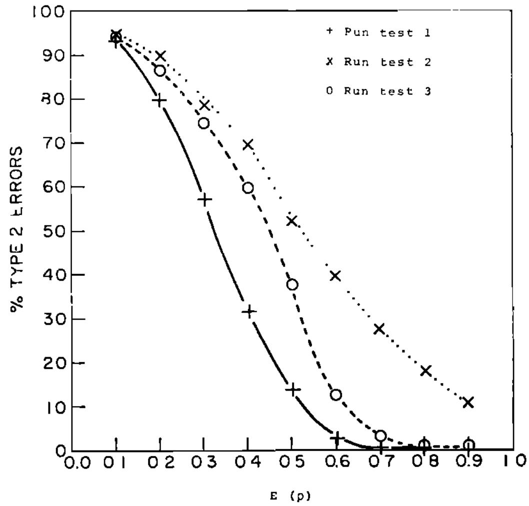

Fig. 11. Power of 'runs' tests, Markov processes, $N = 100$.

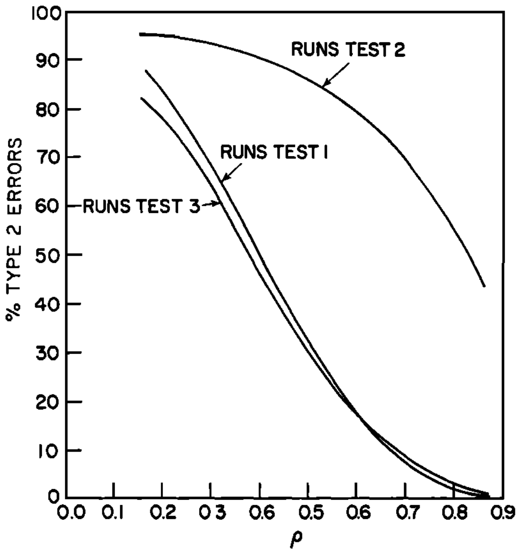

Fig. 12. Power of 'runs' tests, filtered type 2 processes, $N = 100$.

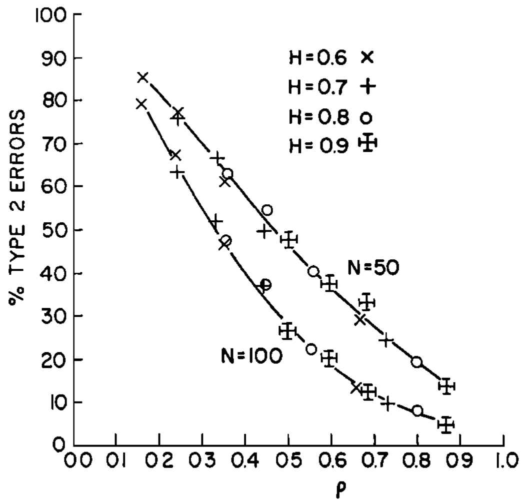

Fig. 14. Power of K test, Markov processes.

test with $N = 100$. This difference, which is shown, increases as N becomes smaller. Test 2 has lower power than either test 1 or test 3, especially for filtered type 2 processes. The three runs tests discussed above are by no means exhaustive, but their low powers relative to Anderson's test suggests that the powers of other runs tests should be examined before they are applied to detect serial dependence.

test tends to yield a distorted view of hydrology, namely, a general absence of long-term persistence. The evident weakness of the Anderson test when confronted with small samples can be partially accounted for by the bias in $\hat{\rho}$ (see later section of this paper).

For the three runs tests, β versus ρ for $N = 100$ is shown in Figures 11 and 12 for Markovian and filtered type 2 processes. For both types of processes, β for given values of ρ are larger for each of the runs tests than for the Anderson

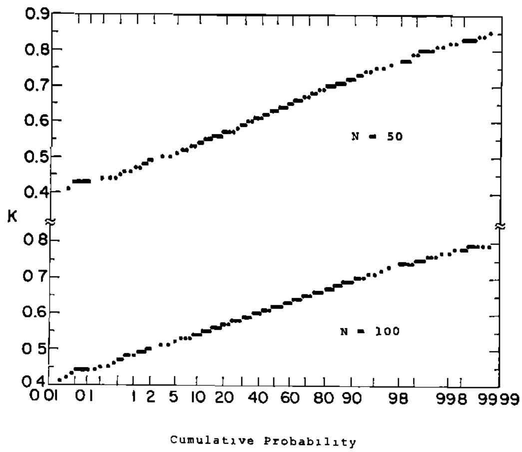

Fig. 13. Distribution functions of K for Gaussian independent processes.

The estimates of h, K, and H may also be used to test for serial dependence. Such a test is developed and shown for K. For Gaussian independent sequences, the distribution functions for K were empirically developed from 10,000 generated sequences for $N = 50$ and for $N = 100$. From these distribution functions, shown in Figure 13, confidence limits may be obtained to correspond to values of α.

Figures 14 and 15 show β versus ρ for the K test, where $\alpha = 0.05$, for $N = 50$ and 100 for Markovian and filtered type 2 processes. This test has lower power when applied to sequences generated by filtered type 2 processes than by Markov processes. Also, for both types of processes, the power of the K test is less than that for the Anderson test. Whereas a test based on individual estimates of K is not useful, the difference between sets of observed K and those for the population of K for Gaussian independent series of equal length is a powerful test for independence. For instance, for randomly selected sets of nonoverlapping sequences, $N = 50$, of filtered type 2 fractional noise, $h = 0.6$, $M = 20{,}000$, $p = 20$, we found that sets of 10 were sufficient to separate out serial dependent sequences from independent ones 75% of the time at the 5% level. With sets of 20 the type 2 errors fell to zero. In hydrology we cannot make a random selection of time periods, and the sets of observed K are likely to be based on series that are cross correlated. Limited work has indicated that cross correlation must be in excess of 0.6 before a test of mean K loses power. Further discussion of the application of the mean K test to hydrology would take us into the very real problems of regionalization, and we shall defer this until another time.

BIAS IN $\hat{\rho}$

Apart from the large type 2 errors associated with test of serial dependence, the tests are further hampered by the fact that $E(\hat{\rho}) < \rho$, where $\hat{\rho}$ is given by (12). The magnitude of the bias in $\hat{\rho}$ is dependent on the magnitudes of N and ρ and the generating process. The bias is depicted graphically in Figures 16, 17, and 18. From these figures it is seen that the bias is less for Markovian processes than for filtered type 2 processes.

The biases were determined by generating sequences with a given value of ρ and for each sequence determining $\hat{\rho}$ by (12). For the Markov process, 4000, 2000, and 1000 sequences of lengths $N = 25$, 50, and 100, respectively, were generated for each value of $\rho = 0.1(0.1)0.9$. For the filtered type 2 process, the bias was determined from five sequences each of length 10,000 for each value of N — 25, 50, and 10) and for each value of $h = 0.6$ and 0.9 with $p = 1$, 5, 10, and 20 and $M/p = 1000$. From Figures 16, 17, and 18 it is evident that the small sample bias in $\hat{\rho}$ is much more extreme in the filtered fractional worlds than in the Markovian; in addition it seems reasonable to expect the bias in $\hat{\rho}$ to increase with $M_{(N,p,h)}$.

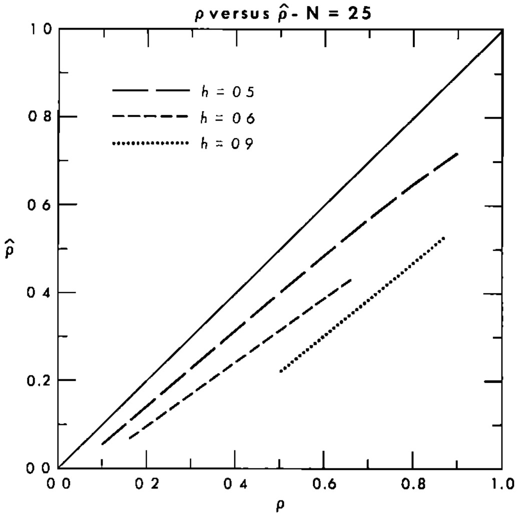

Fig. 16. ρ versus $\hat{\rho}$ — $N = 25$.

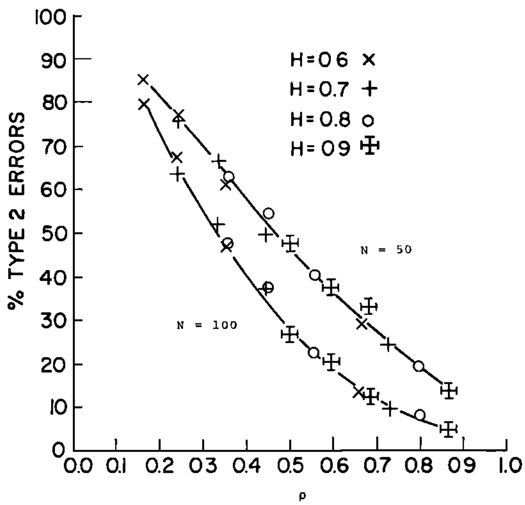

Fig. 15. Power of K test, filtered type 2 processes.

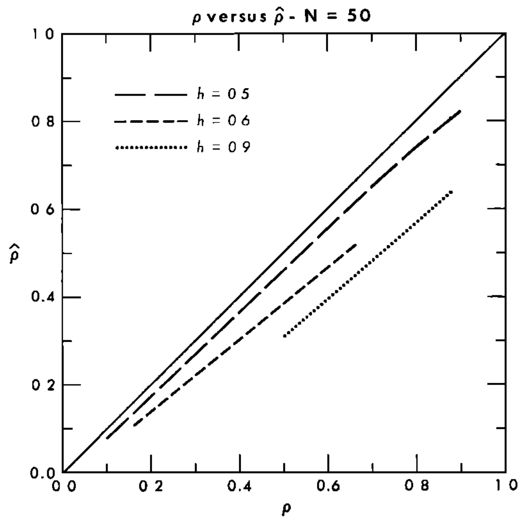

Fig. 17. ρ versus $\hat{\rho}$ — $N = 50$.

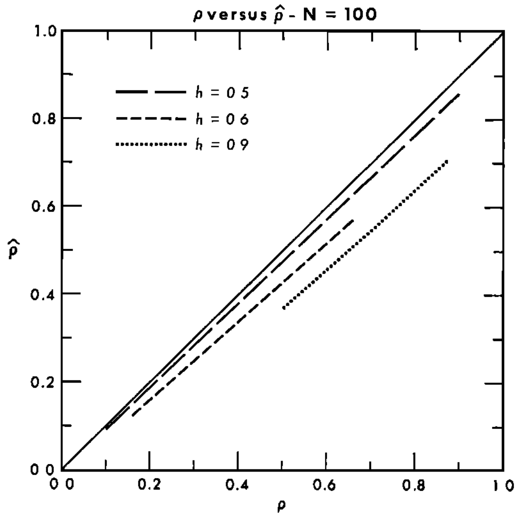

Fig. 18. ρ versus $\hat{\rho}$ $-$ N = 100.

In generating synthetic sequences, the observed value $\hat{\rho}$ assumes the role of ρ in the generating process. For an infinite synthetic sequence, $\rho(s) = \hat{\rho}$, where $\rho(s)$ is the lag-one serial correlation coefficient for the synthetic sequence. In water resource systems planning, the length of the synthetic sequence would correspond to the system's economic time horizon, which is usually taken to be $N = 50$ or 100. Several sequences of length N would be used to evaluate the system's response to streamflow input. A synthetic sequence of length N will tend to yield estimates of $\rho(s)$ that are biased. That is, $\tilde{\rho}(s) < \rho(s)$, where $\tilde{\rho}(s)$ is the lag-one serial correlation for the synthetic sequence of length N.

If ρ denotes the underlying lag-one serial correlation for a sequence, then

$$\rho > \hat{\rho} = \rho(s) > \hat{\rho}(s) \tag{18}$$

Thus the sequences used in systems planning will tend to be more nearly independent than those generated by nature. Depending on N and ρ, the short synthetic sequences may fail to reflect the sensitivity of systems planning to serial correlation. This failure can be illustrated by an example. Suppose streamflow is generated by a Markov process with $\rho = 0.4$. Given an observed sequence of length $N = 25$, the expected value of $\hat{\rho}$ will be 0.3. If a Markov process with $\hat{\rho} = 0.3$ is used to generate synthetic sequences of length $N = 25$, then for these sequences $\tilde{\rho}(s)$ will tend to be about 0.2. If, however, the generating process is a filtered type 2 process with $h = 0.6$, $M = 5000$, and $p = 5$, then $\rho = 0.35$, and for $N = 25$, $\hat{\rho} = 0.2$, and $\tilde{\rho}(s) =$ 0.14.

Because of bias in estimating ρ, a choice must be made as to whether ρ or $\hat{\rho}$ is to be preserved in synthetic sequences of length N. If ρ is to be preserved, then it is necessary to assume a value $\rho' > \rho$. Similarly if $\hat{\rho}$ is to be preserved, then a value $\rho'' > \hat{\rho}$ must be assumed. The choice of the values of ρ' and ρ'' depends on N, the observed value of $\hat{\rho}$, and the assumed generating process.

CONCLUSIONS

Correlograms for hydrologic sequences are characterized by a large variability in the estimates of $\rho(u)$, particularly for u large. This variability is ascribed to sampling errors whose magnitudes vary inversely with N, where N is the sequence length. Because of this variability, it is difficult to infer the sequence generating process from a correlogram.

Quite often, correlograms yield significant values of $\hat{\rho}_u$ for u large. The tendency has been to write off such values as chance events or to seek a cyclic explanation rather than to consider them as symptomatic of long-term persistence. The search for a cyclic explanation has been hampered by the instability of the observed wavelengths among different records, the lack of physical models, and the unknown power of statistical cycle tests when in the presence of long-term persistence.

Markovian and filtered type 2 fractional noise processes yield correlograms that have distinctly different sampling properties. Sequences generated by the latter process and hydrologic sequences that have identical values of N, $\hat{\rho}$, and K are inseparable from each other. That is, hydrologic sequences exhibit long-term persistence. Various tests of serial independence fail to confirm the visual evidence offered by correlograms. The reason for this failure appears to be that for values of N, $\hat{\rho}$, and K characteristic of hydrologic sequences, the tests have small power, particularly those tests based on the theory of runs.

Estimates of ρ are biased downward. Though the bias decreases with N, it is greater for filtered type 2 processes than for Markovian processes. This bias is carried over into generated synthetic sequences. Synthetic sequences of length N tend to yield estimates of serial correlation that are smaller than that observed for an historical sequence which is itself a downward biased estimator of ρ.

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