Computer Experiments with Fractional Gaussian Noises. Part 3, Mathematical Appendix

BENOIT B. MANDELBROT

AND

JAMES R. WALLIS

International Business Machines Research Center
Yorktown Heights, New York 10598

The present 'Appendix' is devoted to mathematical considerations designed to fill in the innumerable logical gaps left in the preceding paper. Even though most proofs will be merely sketched or omitted, the notation remains heavy, and some readers may wish to skip to formulas 1 and 2, which define 'Type 1 functions.'

Most of what follows is closely patterned after *Mandelbrot and Van Ness* [1968], which contains additional mathematical details and references to the few earlier articles, where fractional Gaussian noises were fleetingly considered as purely mathematical entities.

BROWNIAN MOTION: DEFINITION, SELF-SIMILARITY, AND THE $\sqrt{s}$ LAWS

First, some properties of the Brownian motion random process, designated $B(t)$, also called 'Bachelier process,' or 'Wiener process.' The main property of $B(t)$ is that, for every $\epsilon > 0$, the sequence of increments $B(t + \epsilon) - B(t)$ (defined for t multiple of ϵ) is a sequence of independent Gauss random variables with zero mean and variance equal to ϵ. Conversely, $B(t + \epsilon)$ (considered for t multiple of ϵ) is the cumulative sum of a sequence of independent Gauss random variables. The function $B(t)$ is continuous (actually, 'almost surely continuous,' but we may disregard this technicality). Therefore, $B(t)$ can be considered as a tool for interpolating a function of discrete time (namely, the above mentioned cumulative sum of independent Gauss random variables) into a function of continuous time. The sequence $B(t + \epsilon) - B(t)$ itself is called a 'discrete time Gaussian white noise.'

Brownian motion is a 'self-similar process' and satisfies '$\sqrt{s}$ laws.' To explain these terms, suppose that time units are divided by the ratio r, assumed to be positive but otherwise arbitrary. The new time u and the old time t are related by $t = ru$. Then, a fundamental property of Brownian motion is that the functions $B(t) - B(0) = B(ru) - B(0)$ and $r^{0.5}[B(u) - B(0)]$ are generated by the same probabilistic mechanism. Such functions are said to be 'identical in distribution.' In other words, by changing time in the ratio r and the original function $B(t) - B(0)$ in the ratio $r^{0.5}$, one obtains a rescaled function identical in distribution to the original $B(t) - B(0)$.

A *first corollary* of this self-similarity is that the variance of $[B(t + s) - B(t)]/\sqrt{s}$ is equal to 1. Thus, we obtain

> Standard deviation of $B(t + s) - B(t) = \sqrt{s}$ for every t and s.

This is called a 'uniform $\sqrt{s}$ law for the standard deviation.' (The term 'uniform' stands for 'valid for every t and every s.')

A *second corollary* of self-similarity concerns the population range defined as

$$R_P(t, s) = \max_{0 \le u \le s} [B(t + u) - B(t)] - \min_{0 \le u \le s} [B(t + u) - B(t)]$$

where the max and the min are taken by *continuously* varying the 'dummy variable' u. (To justify our calling R_P a range, and to proceed

to speak of the range $R(t, s)$, it will be necessary to write $B(t) - B(0)$ as $\sum_{u=1}^{t} X(u)$, with $\mathcal{E}X = 0$. This will be done below.) We have for R_P the property that

The distribution of the random variable $R_P(t, s)/\sqrt{s}$ is independent of t and s.

This is a 'uniform $\sqrt{s}$ law for the population range.' The moments of the random variable $R_P(t, s)$ have been evaluated by *W. Feller* [1951].

A *third corollary* of self-similarity is that

The distribution of the random variable $R(t, s)/\sqrt{s}$ is independent of t and s.

The moments of $R(t, s)$ have also been evaluated by *W. Feller* [1951], who proved that $\mathcal{E}[R(t, s)/\sqrt{s}] < \mathcal{E}[R_P(t, s)/\sqrt{s}]$, and that $R(t, s)$ is less scattered around its expectation than $R_P(t, s)$.

FRACTIONAL BROWNIAN MOTION: DEFINITIONS, SELF-SIMILARITY, AND THE s^H LAWS

From the mathematical viewpoint, fractional Brownian motions with $H \neq 0.5$ are characterized by three properties: (a) They are Gaussian processes; (b) Their increments $B_H(t) - B_H(t - \epsilon)$ constitute a stationary random process, with t multiple of ϵ; (c) They are self-similar, in the sense that, if time is changed in the ratio r, and the function $B_H(t) - B_H(0)$ is changed in the ratio r^H, one obtains a function identical in distribution to the original $B_H(t) - B_H(0)$.

It can be shown that such a function $B_H(t) - B_H(0)$ exists if and only if $0 < H < 1$, and that $B_H(t)$ can be deduced from $B(t)$ by forming the integral

$$B_H(t) - B_H(0) = \int_{-\infty}^{0} [(t-u)^{H-0.5} - (-u)^{H-0.5}]\, dB(u) + \int_{0}^{t} (t-u)^{H-0.5}\, dB(u)$$

This integral transforms all the values of the process $B'(t)$ corresponding to all instants u prior to t, by weighting and then adding them. The result can be shown to be a continuous function of t (actually, 'almost surely continuous,' but we may again disregard this technicality). The above construction may appear artificial. To make it symmetric, it is tempting to write for every t' and t''

$$B_H(t') - B_H(t'') = \int_{-\infty}^{t'} (t'-u)^{H-0.5}\, dB(u) - \int_{-\infty}^{t''} (t''-u)^{H-0.5}\, dB(u)$$

Unfortunately, both integrals on the right-hand side are divergent, so that this way of writing is mathematically incorrect. It is mainly useful as a mnemonic device in that it suggests that fractional Brownian motion is formed by weighting past values of a white noise by $(t-u)^{H-0.5}$.

It is clear that fractional Brownian motion with $H = 0.5$ simply reduces to

$$B_{0.5}(t) - B_{0.5}(0) = \int_{0}^{t} dB(u) = B(t) - B(0)$$

Thus, $B_{0.5}(t)$ is ordinary Brownian motion and, as announced, fractional Brownian motion is a generalization of ordinary Brownian motion to values of H different from $H = 0.5$.

Fractional Brownian motion is a self-similar process and satisfies an 's^H law.' That is, if time is changed in the ratio r, and the original fraction $B_H(t) - B_H(0)$ is changed in the ratio r^H, one obtains a rescaled fraction identical in distribution to the original $B_H(t) - B_H(0)$.

A *first corollary* of self-similarity is that the variance of $s^{-H}[B_H(t+s) - B_H(t)]$ is independent of t and s. Thus

$$\mathcal{E}[B_H(t+s) - B_H(t)]^2 = \mathcal{E}[\Delta B_H]^2 = C_H s^{2H} \quad \text{for every } t \text{ and } s$$

Here the constant C_H is given by the integral

$$C_H = \mathcal{E}[B_H(t+1) - B_H(t)]^2$$

which one can show to equal

$$\int_0^1 v^{2H-1}\, dv + \int_1^\infty [v^{H-0.5} - (v-1)^{H-0.5}]^2\, dv$$

In other terms

The standard deviation of $B_H(t+s) - B_H(t) = \sqrt{C_H} s^H$ for every t and s.

Of these two equivalent statements, the first was used in the discussion of 'terms' in Part 1. The second is a 'uniform s^H law for the standard deviation,' to be used in discussing the corollary to Hurst's law due to *Langbein* [1956].

A *second corollary* of self-similarity concerns the population range $R_P(t,s)$. One has

The distribution of the random variable $R_P(t, s)\ s^{-H}$ is independent of t and s.

This is a 'uniform s^H law for the population range.' In particular, D_H being a constant different from C_H, one has

$$\mathcal{E}[R_P(t, s)] = D_H s^H \quad \text{for every } t \text{ and } s$$

A *third corollary* of self-similarity concerns the sample sequential range $R(t, s)$. One has

The distribution of the random variable $R(t, s)\ s^{-H}$ is independent of t and s.

It appears most plausible that, just as for $H = 0.5$, one has $\mathcal{E}[R(t, s)\ s^{-H}] < \mathcal{E}[R_P(t, s)\ s^{-H}]$ and that $R(t, s)$ is less scattered around its expectation than $R_P(t, s)$.

DISCRETE-TIME FRACTIONAL NOISES: DEFINITIONS

The sequence of increments of $B_H(t)$, namely the sequence of values of $\Delta B_H(t) = B_H(t) - B_H(t - 1)$, with t an integer, is called 'discrete-time fractional noise.' It can be deduced from a Brownian motion $B(u)$ by the formula

$$\Delta B_H(t) = B_H(t) - B_H(t - 1) = \int_{-\infty}^{t} K_H(t - u)\, dB(u)$$

with the 'kernel function' $K_H(u)$ given by

for $0 < u < 1$, $K_H(u) = u^{H-0.5}$
for $u > 1$, $K_H(u) = [u^{H-0.5} - (u - 1)^{H-0.5}]$

For $H = 0.5$, $\Delta B_H(t)$ reduces to a discrete time Gaussian white noise with $\epsilon = 1$. The covariance of $\Delta B_H(t)$, that is $\mathcal{E}[\Delta B_H(t) \cdot \Delta B_H(t + s)]$, takes the following form:

If $s = 0$, $\mathcal{E}[\Delta B_H(t)\ \Delta B_H(t + s)] = \mathcal{E}[\Delta B_H{}^2] = C_H s^{2H}$

If $s > 0$,

$$\begin{aligned}
&\mathcal{E}[\Delta B_H(t)\ \Delta B_H(t + s)] \\
&= \mathcal{E}[\Delta B_H(1)\ \Delta B_H(s + 1)] \\
&= (1/2)\{\mathcal{E}[\Delta B_H(1) + \cdots \Delta B_H(s + 1)]^2 \\
&\quad + \mathcal{E}[\Delta B_H(2) + \cdots \Delta B_H(s)]^2 \\
&\quad - \mathcal{E}[\Delta B_H(1) + \cdots \Delta B_H(s)]^2 \\
&\quad - \mathcal{E}[\Delta B_H(2) + \cdots \Delta B_H(s + 1)]^2\} \\
&= (1/2)\{\mathcal{E}[B_H(s + 1) - B_H(0)]^2 \\
&\quad + \mathcal{E}[B_H(s) - B_H(1)]^2 - \mathcal{E}[B_H(s) \\
&\quad - B_H(0)]^2 - \mathcal{E}[B_H(s + 1) - B_H(1)]^2\} \\
&= (C_H/2)[(s + 1)^{2H} - 2s^{2H} + (s - 1)^{2H}]
\end{aligned}$$

Clearly, if $X(t) = \Delta B_H(t)$, one has $X^*(t) = \sum_{u=1}^{t} [B_H(u) - B_H(u - 1)] = B_H(t) - B_H(0)$. Thus, one has

Standard deviation of $X^*(t + s) - X^*(t) = \sqrt{C_H} s^H$ for all *integer* values of t and s.

This is a form of 's^H law for the standard deviation,' but it is no longer uniformly valid, but only valid for integer t and s.

In the section after next, we shall encounter additional complications in searching to identify an s^H law applicable to the range of discrete time fractional noise.

CONTINUOUS TIME WHITE OR FRACTIONAL NOISES

It is tempting to try to work with the derivatives $B'(t)$ or $B_H'(t)$, to be called, respectively, 'continuous time white noise' or 'continuous time fractional noise.' Extraordinary care is, however, required in handling $B'(t)$ or $B_H'(t)$. The reason is that, even though $B_H(t)$ is a continuous function of t, its local behavior is extremely irregular. It may be recalled that the derivative of a function $X(t)$ at the point t is defined as the limit for $\delta \to 0$ (if such a limit exists) of the finite local slope defined by $\delta^{-1}[X(t + \delta) - X(t)]$. For fractional Brownian motion, however, the local irregularity is such that $\delta^{-1}[B_H(t + \delta) - B_H(t)]$ *does not tend towards any limit* as $\delta \to 0$. Thus, $B_H(t)$ is continuous but has no derivative in the ordinary sense. Perhaps surprisingly, this seemingly 'pathological' feature is needed to make $B_H(t)$ a realistic model of geophysical phenomena. Total precipitation between times 0 and t, for example, must be a continuous function of t, and such is indeed the case for $B_H(t)$. But the 'instantaneous' precipitation is extraordinarily variable in its detail. One would not even want to follow it in continuous time, except after appropriate smoothing. In fact, if a model suggests that instantaneous precipitation can be followed, it is unreasonable. To model instantaneous precipitation by the formal derivative $B_H'(t)$ is acceptable precisely *because* this

$B_H'(t)$ becomes mathematically meaningful only after it is agreed that it must always be examined through some smoothing mechanism to eliminate high-frequency jitter. The mathematical phrase for this fundamental issue is that $B_H'(t)$ cannot be considered a random function, but is rather a random Schwartz distribution. As long as the questions asked are physically meaningful, the technicalities of the Schwartz theory will not affect us.

COMPARISON OF THE POPULATION RANGES FOR DISCRETE- AND CONTINUOUS-TIME FRACTIONAL NOISES: GRID CORRECTION

By the definitions of the last two sections

$$\int_0^t B_H'(u)\,du = \sum_{u=1}^{t} [B_H(u) - B_H(u-1)]$$

$$= B_H(t) - B_H(0)$$

Thus, both for discrete and for continuous time, the population range of fractional noise is yielded by the following formula:

$$R_P(t, s) = \max_{0\le u\le s} [B_H(t+u) - B_H(t)] - \min_{0\le u\le s} [B_H(t+u) - B_H(t)]$$

The difference is that, when time is discrete, one deals with expressions, to be called 'grid max,' 'grid min,' and 'grid R_P,' that are obtained by comparing $s + 1$ quantities. When time is continuous, one deals with expressions, to be called 'true max,' 'true min,' and 'true R_P,' that are obtained by comparing an infinite number of quantities, among which the above $s + 1$ are included. Visualize the program that selects max, min, or R_P as a 'comparison shopper.' When time is made continuous, the selection open to our shopper is made wider, so that the 'max' he will select will be at least as large (and possibly larger) than the 'max' he would have previously selected. Similarly, the 'min' will be at least as small and possibly smaller, and the difference between 'max' and 'min' will be as large as before or possibly larger. We thus have the inequalities

$$\text{'true max'} - \text{'grid max'} \ge 0$$

$$\text{'true min'} - \text{'grid min'} \le 0$$

$$\text{'true } R_P\text{'} - \text{'grid } R_P\text{'} \ge 0$$

To restate the first of these results differently, let us designate by $t + u_{\max}$ the precise instant (in discrete time) where $B_H(t + u) - B_H(t)$ attains its 'grid max.' Assuming in addition that $s > 2$, $1 \le u_{\max} \le s - 1$, the above definition implies that $B_H(t + u_{\max}) > B_H(t + u_{\max} - 1)$ and that $B_H(t + u_{\max}) > B_H(t + u_{\max} + 1)$. Moreover, it would be extraordinarily unlikely for $B_H(t + u) - B_H(t)$ to have a sharp peak at the precise instant $t + u_{\max}$. It follows that $B_H(t + u) - B_H(t)$ has, somewhere in the interval from $t + u_{\max} - 1$ to $t + u_{\max} + 1$, a local maximum higher than $B_H(t + u_{\max}) - B_H(t)$. Since 'true max' is at least equal to that local maximum, we have proved that 'true max' − 'grid max' ≥ 0. In fact, one can prove much more. Except when s is very small, say for $s \ge 3$, the distributions of the differences 'true max' − 'grid max' and 'true R_P' − 'grid R_P' will depend very little upon s. Since R_P tends to increase with s, the relative error due to the grid, defined as 'true R_P'/'grid R_P' − 1, will tend to decrease as $s \to \infty$. For large s, this relative grid error will be very small.

Moreover, as s increases from 1 to ∞, $\mathcal{E}[R_P(t, s)s^{-H}]$ will increase from 0 to the value D_H that enters into the s^H law for the range of fractional Brownian motion in continuous time. Thus, using doubly logarithmic coordinates, $\mathcal{E}[R_P(t, s)]$ will plot as a straight line of slope H preceded by a portion with strong downward concavity. (We have checked that such is indeed the case, but the corresponding figures have not been reproduced for lack of space.)

The precise position of the bend between this initial transient and the asymptotic s^H behavior is hard to determine precisely by analysis. Had this bend turned out to be positioned at an extremely large value of s, the s^H asymptote would have had no relevance to modeling. Our computer experiments have demonstrated that, even for simple approximations to fractional noise, this bend is positioned at a small value of s.

The need for a grid correction for small s, and the resulting complicated transient, diminish the beauty of the theory. The size of the grid correction could readily be decreased by selecting a finer grid for u. But these would be no practical point in such formal 'beautification.' After all, we are working out a model of the physical reality, and we know that

records are necessarily taken in discrete time. In order that the theory be comparable with data, the theory must also use a discrete grid, and units of time can be chosen so that the theoretical and practical grids give identical results.

THE SAMPLE RANGE AND THE RESCALED SAMPLE RANGE

The study of the sample range $R(t, s)$ is a bit more complicated than that of R_P and will not be given here. The study of the ratio $R(t, s)/S(t, s)$ is even more complicated. Suffice it to observe that there is a partial compensation between the grid correction and the correction due to division by $S(t, s)$. As a result, the ratio $R(t, s)/S(t, s)$ has a smaller initial transient than the expression R_P studied above.

DEFINITION OF TYPE 1 APPROXIMATE FRACTIONAL NOISE

The integral $\int_{-\infty}^{t} K_H(t - u)dB(u)$, which defines $B_H(t) - B_H(t - 1)$, cannot be evaluated exactly on a computer. Its numerical evaluation necessarily involves three approximations:

1. The span $-\infty < u < t$ must be replaced by finite span $-M + t < u < t$. This introduces a *low frequency* error term. Its most striking consequence is to make the asymptotic behavior of $R(t, s)/S(t, s)$ follow an $\sqrt{s}$ law rather than the s^H law.

2. A discrete grid ϵ must be selected for the variable of integration u; in particular, the infinitesimal $dB(u)$ must be replaced by a finite difference $B(u + \epsilon) - B(u)$. This introduces a *high frequency* error term.

3. Once the grid $\epsilon > 0$ is selected, there is little point in computing the kernel K_H exactly. Simplification of the analytic form of K_H need not introduce major additional errors.

4. Finally, another discrete grid must be selected for $K(t - u)[B(u + \epsilon) - B(u)]$, the values of this integrand being computed to a finite number of decimals.

Two typical approximations to $B_H(t)$, satisfying the above conditions, will be described in the present and next sections.

In our first and foremost approximation to fractional noise, leading to Type 1 functions, step (3) is *not* taken, whereas steps (1) and (2) are taken by selecting $\epsilon = 0.10$ and replacing the continually varying kernel $K_H(u)$ by a stepwise varying kernel $K_1(u|H, M)$, illustrated on Figure 1 and described as follows:

When $u > M$, $K_1(u \mid H, M) = 0$

When $u < M$ and u is an exact multiple of $1/10$, $K_1(u \mid H, M) = K_H(u)$.

When $u < M$ and u is not an exact multiple of $1/10$, and if w is the smallest multiple of $1/10$ whose value exceeds u, $K_1(u \mid H, M) = K_H(w)$

Using this kernel, one can define the function

$$F_1(t \mid H, M) = \int_{-0}^{\infty} K_1(t - u \mid H, M)\, dB(u) \qquad (1)$$

to be called a Type 1 approximation.

To consider the high-frequency aspects of the function F_1, set $M = \infty$. The kernel $K_1(u|H, \infty)$ is a good approximation to $K_H(u)$ when H is such that $\Delta B_H(t)$ does not have very strong high-frequency components. Such is the case when $K_H(u)$ is a continuous function of u, that is, if $0.5 < H < 1$. K_1 is a less good approximation when H is such that $\Delta B_H(t)$ has very strong high-frequency components. Such is the case when $K_H(u)$ is discontinuous, that is, if $0 < H < 0.5$. Generally, the approximation of $K_H(u)$ by $K_1(u|H, \infty)$ improves as H increases from 0 to 1. This is precisely in line with the results of our experimental study of the behavior of $R(t, s)/S(t, s)$ for Type 1 functions.

There is, of course, nothing sacred about the value 0.10 for ϵ. If this value is replaced by a finer grid, the approximation of $K_H(u)$ by $K_1(u|H, \infty)$ is improved in all values of H, but particularly for small H. In other words, to achieve the same preset quality of approximation for every value of H, it is necessary to make ϵ a function of H, with $\epsilon(H)$ decreasing as H increases from 0 to 1.

The additional low-frequency error introduced by setting $M < \infty$ is essentially independent of ϵ.

A moving average representation. While our kernel K_1 is stepwise varying, the function F_1 is to be evaluated only when t is an integer.

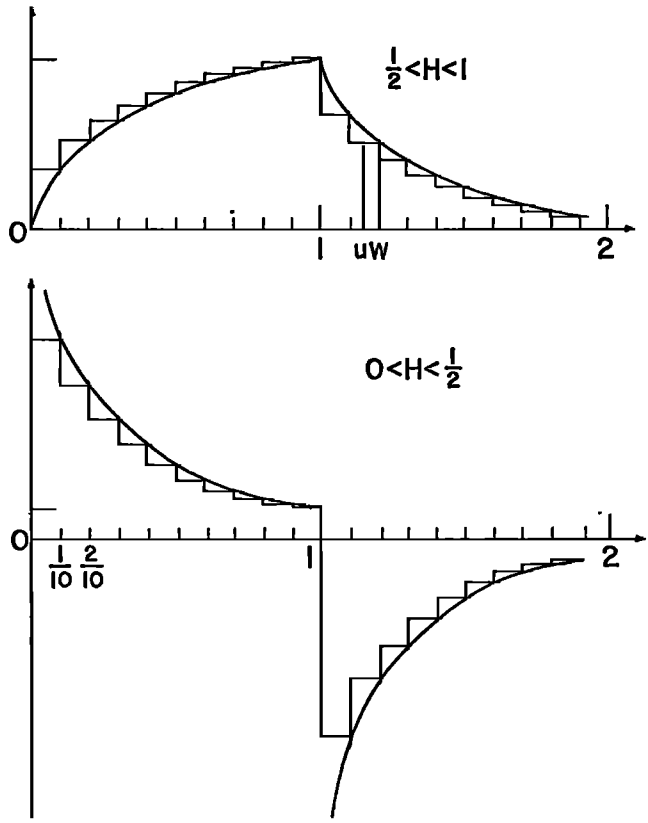

Fig. 1. Relation between the kernels K_H and K_1. On both portions of the figure, the bold lines represent the overall shapes of the kernel $K_H(u)$. The thin lines illustrate the definition of w, and of the stepwise varying approximating kernel $K_1(u|H, M)$. 'Staircase-shaped' curves were drawn, but the risers were inserted solely for legibility; only the 'treads' count. These step functions correspond to representations of the relevant random functions by moving average *integrals* in continuous time. To obtain the representations by moving average *sums*, one should replace each tread by its point farthest to the right.

For such values of time, our Type 1 function can be rewritten as a discrete time moving average of the form

$$F_1(t \mid H, M) = \sum_{n=10(t-M)}^{10t-1} K_1(t - n/10 \mid H, M)G(n/10)$$

In this formula, the kernel K_1 has the same analytic form as the kernel K_1 defined previously; but of course the argument in the kernel is now a multiple of 1/10 so that the complications relative to w no longer arise. As to the variables $G(n/10)$, defined for all integer n, they are independent Gaussian with zero mean and variance 1/10.

(It may be observed that in moving average integrals, both the time t and the dummy variable u vary continuously; in moving average sums, it is usual that the time t and the dummy variable are both integers. In the above definition of a Type 1 function, on the contrary, the time t is an integer but the dummy variable is a multiple of 0.1, which puts it somewhere between an integer and a continuously variable quantity.)

DEFINITION OF TYPE 2 APPROXIMATE FRACTIONAL NOISE

Type 1 functions require quite extensive computations. To simplify them, one could select a rougher grid, say ϵ equal to 1, while

also replacing $K_H(u)$ by some more readily manageable function (step (3) of the approximations listed in the section before last). The latter aim may be fulfilled by observing that, for large u, $K_1(u|H, \infty) \sim (H - 0.5)\ u^{H-1.5}$. The function $u^{H-1.5}$ may be chosen as a simplification for K_1 for large u. But this function unfortunately becomes infinite for $u = 0$, so that a different analytic formula must be selected near $u = 0$. Different approaches must be followed when $0 < H < 0.5$ and when $0.5 < H < 1$.

For $0.5 < H < 1$ we shall define a Type 2 approximating kernel $K_2(u|H, M)$ as follows:

If $u > M$, $K_2(u \mid H, M) = 0$

If $u < M$ and u is an integer, $K_2(u \mid H, M) = (H - 0.5)\ u^{H-1.5}$

If $u < M$ and u is not an integer, and if w is the smaller integer whose value exceeds u,

$$K_2(u \mid H, M) \sim (H - 0.5)\ w^{H-1.5}$$

The discrete time moving average corresponding to this kernel takes the form

$$F_2(t \mid H, M) = (H - 0.5) \sum_{u=t-M}^{t-1} (t - u)^{H-1.5} G(u) \tag{2}$$

which is recognized as the Type 2 approximation already written down at the very beginning of this paper.

Type 1 and Type 2 functions share essentially identical very low frequency properties but have different high frequency properties. Therefore, Type 1 approximations may be expected to be best when high frequency effects are slight, that is, when H is near 1. We saw in our discussion of R/S that the two approximations indeed differ little for $H = 0.9$. When, however, H reaches $H = 0.5$, the kernel K_1 vanishes except for $u < 1$, whereas K_2 remains nonzero throughout $u < M$. The two approximations therefore differ greatly for $H = 0.5$, and the approximation K_2 is of limited scope. Its usefulness is due to its formal simplicity, combined with the fact that empirical values of H are often near 1.

For $0 < H < 0.5$, high-frequency effects are very strong, as we well know (see Figure 17 of *Mandelbrot and Wallis* [1969*b*]). This makes Type 2 approximations a bit more complicated to implement. In writing the approximate kernel K_2, one must ensure that $\Sigma_{u=0}^{\infty} K_2(u|H, M)$ either vanishes or is very close to zero. This is the purpose behind the positive factor $Q_H G(0)$ introduced in the formula at the beginning of this paper.

NOTATION

The following symbols are used in this and the preceding two papers:

a_f^2	= squared Fourier modulus;
$B(t)$	= Brownian motion random process;
$B_H(t)$	= fractional Brownian motion random process;
C_H	= the positive constant entering into the s^H law for the standard deviation;
D_H	= the positive constant entering into the s^H law for the expectation of the population range;
ΔB_H	= the increment process of fractional Brownian motion, also called fractional Gaussian noise in discrete time;
Δ_{FP}	= the difference between past and future sample averages of a discrete fractional noise;
$\Delta X^*(t)$	= increment of the cumulated process $X^*(t)$;
ϵ	= size of the grid when time is discrete;
$\mathcal{E}$	= expectation, also called expected value, population mean or first moment;
f^{-2k}	= a spectral density with k an integer for white noise, and k a noninteger fraction for fractional noises;
F	= length of future record;
$F_1(t \mid H, M)$	= Type 1 approximation of fractional Gaussian noise;
$F_2(t \mid H, M)$	= Type 2 approximation of fractional Gaussian noise;
$G(u)$	= sequence of independent Gaussian random variables;
H	= the principal parameter of a fractional noise, measured by the asymptotic slope of the plot of $\log \mathcal{E}(R/S)$ versus $\log s$;
$K_H(u)$	= kernel function serving to define an exact fractional noise of exponent H;
$K_1(u \mid H, M)$	= kernel function serving to define a Type 1 approximate fractional noise of memory M;
$K_2(u \mid H, M)$	= kernel function serving to define a Type 2 approximate fractional noise of memory M;
M	= memory parameter of an approxi-

mate fractional noise, that is, point at which the sum defining such a noise is truncated;

M^* = effective memory, i.e., maximum values of the lag s beyond which the fact that $M < \infty$ is felt. For lag s beyond M^*, the usual $\sqrt{s}$ law applies to R/S;

P = length of past record;

Pr = probability;

Q_H = a constant;

r = ratio by which time is reduced to define self similarity;

$R(t, s)$ = sample sequential range for lag s;

$R_P(t, s)$ = population cumulative range of $X(t)$ for lag s;

s = time lag;

$S^2(t, s)$ = the variance of s values of $X(t + 1) \cdots X(t + s)$ around their sample average;

t = time;

t_P = Student's random variable;

T = total available sample size;

u = 'dummy' variable, that is, the index in a summation or integration. There is no necessary relation between the dummy variables appearing in different equations;

v = 'dummy' variable;

$X(t)$ = a discrete time fractional noise;

$X^*(t)$ = $\sum_{u=1}^{t} X(u)$;

$W(t)$ = spectral density.

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