Reply

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Nash and O'Connor question the value of computing all the values of $h_{opt}(k)$, rather than just the parameters governing an assumed functional form of $h(t)$ when, as in problems of synthesis, a parametric representation of $h_{opt}(k)$ is ultimately desired. The answer to this, we believe, lies in the basic difference between the definitions of $h_{opt}(t)$ and h(t). The definition of $h_{opt}(t)$ recognizes the presence of nonlinearities and measurement errors by requiring that the observed input and output data are only *approximately* linearly related. This controlled allowance of error ensures that the calculated $h_{opt}(t)$ will be both stable and realizable. If a parametric representation of $h_{opt}(t)$ is required, it may be carried out subsequently and with the advantage of knowing the form of the function to be fitted. On the other hand, calculation of $h(t)$ assumes the input and output to be *exactly* linearly related. In particular, the method of moments suggested for this determination by Nash and O'Connor is based upon the applicability of the convolution relation. Since only the first two moments are required to fit their two parameter $h(t)$ function, this method of fitting would seem to contain the degree of approximation necessary to ensure stable, realizable results, yet there is evidence to the contrary. In a study of urban catchments (1) it was found that fitting the 'Nash model' by the method of moments using input and output data very often led to physically unrealistic values of the $h(t)$ parameters, whereas for the same events fitting the Nash $h(t)$ to the derived $h_{opt}(k)$ did not.

We thank Nash and O'Connor for calling attention to our faulty reasoning in the interrelationship of equations 4 through 7. Fortunately, this has no effect on what follows, since the problem remains the solution of equation 10 for $h(k)$ using input-output pairs that are not linearly related.

The question of the area enclosed by $h_{opt}(k)$ is one that deserves some discussion, as the absence of any constraint thereon by the authors is intentional. The main purpose of the paper, as stated in both the Abstract and the Introduction, is the development of a computational method for fitting a general linear model to arbitrary physical systems where the physical interpretation allows only positive kernel functions. The particular system chosen for illustration was the catchment, which is the system most frequently linearized by hydrologists. The use of the term 'unit hydrograph' in our choice of title reflects our desire to relate this work to familiar engineering tasks and is not intended to imply a one-to-one correspondence between our assumptions and those in common engineering use. Consider the following:

Barrera and Perkins [1967] have recently pointed out that the *complete* integral representation of a general lumped, linear, time-variant system of order n is

$$g(t) = \int_0^t h(t, \tau)f(\tau)\, d\tau + \sum_{j=0}^{n-1} c_j \frac{\partial^j h(t, 0)}{\partial \tau^j} \quad (1)$$

where the coefficients $c_j : j = 0, \cdots n - 1$ are associated with the initial conditions $g(0)$, $g^1(0)$,

$\cdots g^{n-1}(0)$. These investigators also show that since equation 1 is general it must include the particular case having both zero initial conditions and, for rainfall-runoff systems, zero loss of mass between input and output. The first of these conditions eliminates the second term of equation 1, whereas the second condition requires

$$\int_{\tau}^{\infty} h(t, \tau)\, dt = 1 \qquad (2)$$

Since the kernel $h(t, \tau)$ in equation 1 is unique, equation 2 must apply *under all conditions associated with the use of* equation 1. In practice, however, engineers usually make further simplifying assumptions beyond the lumping and linearization associated with equation 1. First, it is common to assume stationarity, which allows equation 1 to be written

$$g(t) = \int_0^t h(t - \tau)f(\tau)\, d\tau + \sum_{j=0}^{n-1} c_j \frac{\partial^j h(t - 0)}{\partial \tau^j} \qquad (3)$$

and the kernel constraint given by equation 2 still applies. The next most common simplification, which is almost never stated explicitly by hydrologists, is to require that the initial conditions all be identically zero, leaving the familiar

$$g(t) = \int_0^t h(t - \tau)f(\tau)\, d\tau \qquad (4)$$

This requirement is approximated in conventional unit hydrograph theory by restricting the system to that which transforms rainfall excess into direct runoff. Since we are then dealing with solutions to equation 1, equation 2 still applies. In all of the above the 'system' is taken, again usually without explicit statement, as the *solid* physical features of the catchment, and the presence of any water (surface water at least) signifies an excited state of the system. Since one of the major nonlinearities of unsteady free surface flow results from the dependence of wave propagation velocity upon the instantaneous state of system excitation it is not surprising that the above type of linearization has led to difficulty in finding kernels that are unique over a significant range of initial conditions and event magnitudes. Remember, it is applicability over the *complete range* of circumstances that has been invoked to arrive at equation 2.

These difficulties suggest a perturbation approach to the linearization which is equivalent to assuming that the dry catchment behaves linearly over a range of circumstances surrounding some initial state. The equivalent linear system is thus different for each initial state. The kernel in equation 1 is now unique only for a particular system state, and the argument leading to equation 2 cannot be employed. Incorporating all the relevant c_j into our definition of the system being studied, equation 1 reduces to

$$g(t) = \int_0^t h(t, \tau)f(\tau)\, d\tau \qquad (5)$$

and *equation 2 will apply only for a very special system which is rarely found in practice.*

If the systems are stationary, equation 5 becomes

$$g(t) - g(0) = \int_0^t h_0(t - \tau)f(\tau)\, d\tau \qquad (6)$$

in which $g(t)$ and $f(t)$ are the *gross* streamflow and rainfall, respectively, and $h_0(t)$ signifies that the kernel is evaluated for the system that exists at $t = 0$. Equation 2 need not be satisfied.

This alternative philosophy, rather than the conventional philosophy, was behind the approach under question by Nash and O'Connor.

Contrary to their claim, we *did not* report difficulty in applying the method to complex storms. We *did* state that the optimization should be carried out over an 'event space' that contains a large number of storms (equivalent initial conditions implied), and that this appeared to lead to programming difficulties, due to the size of the linear programming problems. However, the recent work of *Barrera and Perkins* [1967] has shown that

$$\Phi_{fo}(\tau) = \int_{-\infty}^{\infty} h_{opt}(\sigma)\Phi_{ff}(\tau - \sigma)\, d\sigma \quad \tau > 0 \qquad (6)$$

where, if the event space includes N storms

$$\Phi_{fo}(\tau) = \frac{1}{N}\sum_{k=1}^{N}\int_0^{\infty} f_k(s)g_k(s + \tau)\, d\tau \qquad (7)$$

and

$$\Phi_{ff}(\tau) = \frac{1}{N}\sum_{k=1}^{N}\int_0^{\infty} f_k(s)f_k(s + \tau)\, d\tau \qquad (8)$$

which limits the size of the Wiener-Hopf matrix to that of the largest single event. The programming problem is thus eliminated!

The fact that the derived $h_{opt}(t)$ may fail to resemble the simple linear monotone, as was found for the urban catchments analyzed, reinforces our desire to look at the kernel shape before trying to fit it parametrically!

We cannot give a definite answer to the last question regarding the effects of nonlinearity and of data error on the observed variability of $h_{opt}(t)$ for a given catchment. However, there is evidence (1), through failure to obtain significant correlations between parameters of $h_{opt}(t)$ and parameters of the input rainfall, that for urban catchments errors in the data may be largely responsible for the variability. The computational method developed seems less sensitive to data errors than other methods in use. As an illustration, we have used data from urban catchments (1) to compare the parameters n and k of the Nash unit hydrograph as determined (a) by the method of moments from the input and output data (b) from the moments of $h_{opt}(k)$. These are given in Table 1.

The accuracy with which the various kernels fit the system has been calculated (1) using the 'integral-square error' E, where

$$E_i = \frac{[\sum_k [g(k) - g_i^*(k)]^2]^{1/2}}{\sum_k g(k)} \times 100 \qquad (9)$$

in which $g(k)$ is the observed output, and $g_i^*(k)$ is the predicted output using the kernel derived for this event by the ith method. The average of E_i over the same 19 storms is shown in Table 2.

TABLE 1. Sensitivity of Nash Parameters to Method of Computation
(Storms 1 through 19, Catchment N9)

	n		k	
	Mean	Variance	Mean	Variance
(a) Moments of input-output data	3 14	21 20	2 65	15 20
(b) Moments of $h_{opt}(k)$	1 76	0 60	2 90	8.42

TABLE 2. Average Integral–Square Error in Output
(Storms 1 through 19, Catchment N9)

Method	Average Error, $\bar{E}$ in %
1. Using $h_{opt}(t)$	4.3
2. Using Nash $h(t)$ with parameters obtained from moments of $h_{opt}(t)$	11.4
3. Using Nash $h(t)$ with parameters obtained from input and output moments	15.6

We thank Mr. Nash and Mr. O'Connor for their interest in our work and apologize for the delay in replying.

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